Math 115 — First Midterm

Oct 7, 2014

Name: EXAM SOLUTIONS

Instructor: __________ Section: __________

---

1. **Do not open this exam until you are told to do so.**
2. This exam has 11 pages including this cover. There are 12 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
8. Include units in your answer where that is appropriate.
9. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones.
10. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	10	
2	8	
3	9	
4	12	
5	11	
6	12	
7	11	
8	6	
9	4	
10	8	
11	6	
12	3	
Total	100	

1. [10 points] The following table provides some information on the populations of Detroit and Ann Arbor over time.

Year	1970	2000
Ann Arbor Population (in thousands)	100	114
Detroit Population (in thousands)	1514	

Remember to show your work clearly.

a. [3 points] Suppose that between 1950 and 2000 the population of Ann Arbor grew at a constant rate (in thousands of people per year). Find a formula for a function $A(t)$ modeling the population of Ann Arbor (in thousands of people) t years after 1950.

Solution: Since the population grew at a constant rate, $A(t)$ is a linear function. The slope of the graph of $A(t)$ is

$$\frac{A(50)-A(20)}{50-20} = \frac{114-100}{50-20} = \frac{14}{30} = \frac{7}{15} \approx 0.467\,.$$

Since $A(20) = 100$, we find that $A(t) = 100 + \frac{7}{15}(t-20)$.

Answer: $A(t) =$ $100 + \frac{7}{15}(t-20) = \frac{272}{3} + \frac{7}{15}t \approx 90.67 + 0.467t$

b. [5 points] Suppose that between 1950 and 2000, the population of Detroit decreased by 6% every 4 years. Find a formula for an exponential function $D(t)$ modeling the population of Detroit (in thousands of people) t years after 1950.

Solution: Since $D(t)$ is an exponential function, there are constants a and c so that $D(t) = ca^t$. Since the population of Detroit decreased by 6% every four years, $a^4 = 0.94$. Thus $a = (0.94)^{1/4}$ and $D(t) = c(0.94)^{t/4}$

To solve for c, we note that

$$1514 = D(20) = c(0.94)^{20/4} = c(0.94)^5$$

and thus

$$c = \frac{1514}{(0.94)^5} \approx 2062.94\,.$$

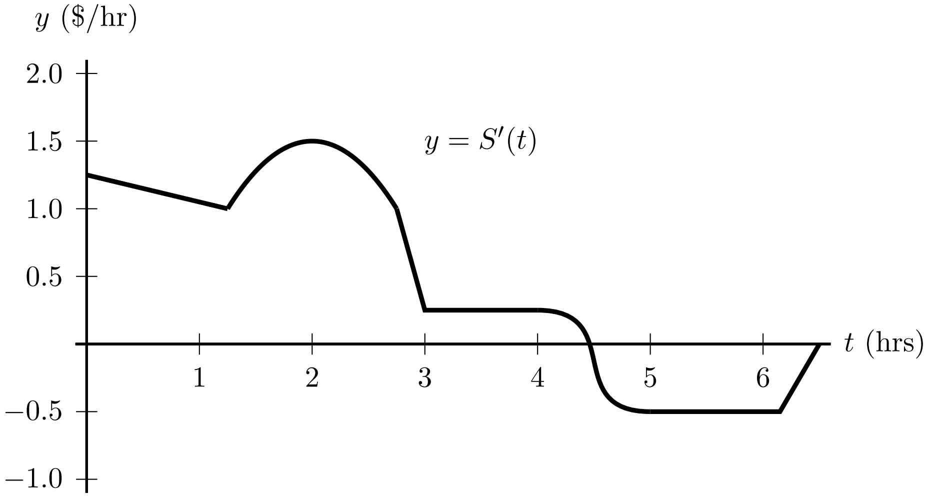
Answer: $D(t) =$ $\frac{1514}{(0.94)^5}(0.94)^{t/4}$ **or** $1514(0.94)^{(t-20)/4}$

c. [2 points] According to your model $D(t)$, what was the population of Detroit in the year 2000? *Include units.*

Solution: $D(50) = \frac{1514}{(0.94)^5}(0.94)^{50/4} \approx 951.89.$

Answer: **about 952 thousand people**

2. [8 points] Suppose that a new company named Calculus Knowledge, which provides calculus consulting work, was posted on the New York Stock Exchange over the summer. Let $S(t)$ be a continuous and differentiable function that models the price, in dollars, of one share of Calculus Knowledge stock t hours after 9:30 am on October 6, 2014.
The graph of $S'(t)$ for $0 \le t \le 6.5$ is shown below.



Note: The graph above is the graph of $S'(t)$. It is **not** the graph of $S(t)$.

a. [2 points] Estimate when the price of the stock is rising most quickly on October 6, 2014.

Answer: **11:30 am**

b. [2 points] According to the model $S(t)$, at which of the times 10 am, 11 am, 12 noon, and 1 pm was the price of one share of Calculus Knowledge stock the <u>lowest</u> on October 6, 2014?
Circle ONE *time or circle* CANNOT BE DETERMINED *if the answer cannot be determined from the information provided.*

[✓ 10 am] 11 am 12 noon 1 pm CANNOT BE DETERMINED

c. [2 points] On which, if any, of the following intervals does it appear that the function $S(t)$ is always decreasing? *Circle* ALL *correct choices or circle* NONE OF THESE *if appropriate.*

$0 < t < 1$ $2 < t < 3$ $4 < t < 5$ [✓ $5 < t < 6$] NONE OF THESE

d. [2 points] On which, if any, of the following intervals does it appear that $S(t)$ is linear? *Circle* ALL *correct choices or circle* NONE OF THESE *if appropriate.*

$0 < t < 1$ $1 < t < 2$ [✓ $3 < t < 4$] [✓ $5 < t < 6$] NONE OF THESE

3. [9 points] Consider the function h defined by

$$h(x) = \begin{cases} \dfrac{60(x^2 - x)}{(x^2+1)(3-x)} & \text{for } x < 2 \\ c & \text{for } x = 2 \\ 5e^{ax} - 1 & \text{for } x > 2 \end{cases}$$

where a and c are constants.

a. [5 points] Find values of a and c so that both of the following conditions hold.

- $\lim\limits_{x\to 2} h(x)$ exists.
- $h(x)$ is not continuous at $x = 2$.

Note that this problem may have more than one correct answer. You only need to find one value of a and one value of c so that both conditions above hold. Remember to show your work clearly.

Solution: In order for $\lim\limits_{x\to 2} h(x)$ to exist, it must be true that $\lim\limits_{x\to 2^-} h(x) = \lim\limits_{x\to 2^+} h(x)$. Now $\lim\limits_{x\to 2^-} h(x) = \dfrac{60(2^2-2)}{(2^2+1)(3-2)} = 24$ and $\lim\limits_{x\to 2^+} h(x) = 5e^{2a} - 1$. So it follows that $5e^{2a} - 1 = 24$. Solving for a, we have

$$\begin{aligned} 5e^{2a} - 1 &= 24 \\ e^{2a} &= 5 \\ 2a &= \ln(5) \\ a &= \ln(5)/2 \approx 0.804. \end{aligned}$$

When $a = \ln(5)/2$, $\lim_{x\to 2} h(x) = 5e^{(\ln(5)/2)*2} = 5e^{\ln(5)} - 1 = 24$. So, h is not continuous at $x = 2$ as long as $\lim_{x\to 2} h(x) \neq h(2)$. Since $h(2) = c$, we can choose c to be any number other than 24.

Answer: $a =$ $\ln(5)/2$ and $c =$ **7 (or any value other than 24)**

b. [2 points] Determine $\lim\limits_{x\to -\infty} h(x)$. If the limit does not exist, write DNE.

Solution: By looking at the rational function

$$\frac{60(x^2-x)}{(x^2+1)(3-x)} = \frac{60(x^2-x)}{-x^3+3x^2-x+3},$$

(the relevant piece of the function here) we see that as $x \to -\infty$, $h(x)$ approaches 0.

Answer: $\lim\limits_{x\to -\infty} h(x) =$ 0

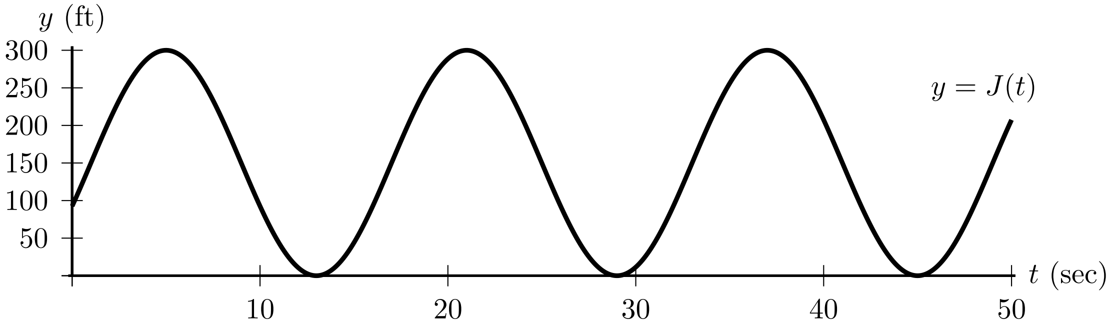
c. [2 points] Find all vertical asymptotes of the graph of $h(x)$. If there are none, write NONE.

Answer: Vertical asymptote(s): NONE

4. [12 points] A dare devil jumps off the side of a bungee jumping platform while attached to a magically elastic bungee cord. Just a few moments after the jump begins, a timer is started and her position is recorded. At t seconds after the timer begins, her distance in feet below the platform is given by the function

$$J(t) = -150\cos(0.125\pi(t+3)) + 150\,.$$

A portion of the graph of $y = J(t)$ is shown below.



Throughout this problem, do not make estimates using the graph.

a. [2 points] Compute the average velocity of the bungee jumper during the first 16 seconds after the timer begins.

Solution: Since $0.125\pi = 2\pi/(\text{period of } J(t))$, the period of $J(t)$ is $2\pi/(0.125\pi) = 16$. Thus, $J(0) = J(16)$ and

$$\text{average velocity} = \frac{J(16) - J(0)}{16} = \frac{0}{16} = 0 \text{ ft/s}\,.$$

Answer: average velocity = 0 **ft/s**

b. [3 points] Recall that *average speed* over an interval of time is given by $\frac{\text{distance traveled}}{\text{time elapsed}}$. Compute the average speed of the bungee jumper during the first 16 seconds after the timer begins.

Solution: Since $J(t)$ has period 16 and amplitude 150,

$$\text{distance traveled during the interval } 0 \le t \le 16 \;= 4(\text{amplitude}) = 600 \text{ ft}$$

Thus, the average speed is $(600 \text{ ft})/(16 \text{ s}) = 37.5 \text{ ft/s}$.

Answer: average speed = 37.5 **ft/s**

c. [5 points] Use the limit definition of instantaneous velocity to write an explicit expression for the instantaneous velocity of the bungee jumper 2 seconds after the timer begins. *Your answer should not involve the letter J. Do not attempt to evaluate or simplify the limit.*

Answer: $\lim_{h\to 0} \frac{-150\cos(0.125\pi(5+h)) + 150 - (-150\cos(0.125\pi(5)) + 150)}{h}$

d. [2 points] Find all values of t in the interval $0 \le t \le 30$ when the instantaneous velocity of the bungee jumper is 0 feet per second.

Solution: The instantaneous velocity is 0 when the tangent line to the position graph is horizontal. This occurs at the maxima and minima on the sinusoidal graph. The first maximum occurs at $t = 5$ and so the first minimum occurs at $t = 13$ (half a period later).

Answer: $5, 13, 21, 29$

5. [11 points] Oren, a Math 115 student, realizes that the more caffeine he consumes, the faster he completes his online homework assignments. Before starting tonight's assignment, he buys a cup of coffee containing a total of 100 milligrams of caffeine.

Let $T(c)$ be the number of minutes it will take Oren to complete tonight's assignment if he consumes c milligrams of caffeine. Suppose that T is continuous and differentiable.

a. [2 points] Circle the ONE sentence below that is best supported by the statement "the more caffeine he consumes, the faster he completes his online homework assignments."

i. $T'(c) \geq 0$ for every value c in the domain of T.

ii. [✓ $T'(c) \leq 0$ for every value c in the domain of T.]

iii. $T'(c) = 0$ for every value c in the domain of T.

b. [1 point] Explain, in the context of this problem, why it is reasonable to assume that $T(c)$ is invertible.

Solution: Since the more caffeine Oren consumes the faster he is able to finish his homework, $T(c)$ is a decreasing function. Thus, $T(c)$ is invertible.

c. [2 points] Interpret the equation $T^{-1}(100) = 45$ in the context of this problem. Use a complete sentence and include units.

Solution: In order for Oren to complete his homework assignment in 100 minutes, he must consume 45 milligrams of caffeine.

d. [3 points] Suppose that p and k are constants. In the equation $T'(p) = k$, what are the units on p and k?

Answer: Units on p are **milligrams of caffeine**

Answer: Units on k are **minutes per milligram of caffeine**

e. [3 points] Which of the statements below is best supported by the equation $(T^{-1})'(20) = -10$? *Circle the* ONE *best answer.*

i. If Oren has consumed 20 milligrams of caffeine, then consuming an additional milligram of caffeine will save him about 10 minutes on tonight's assignment.

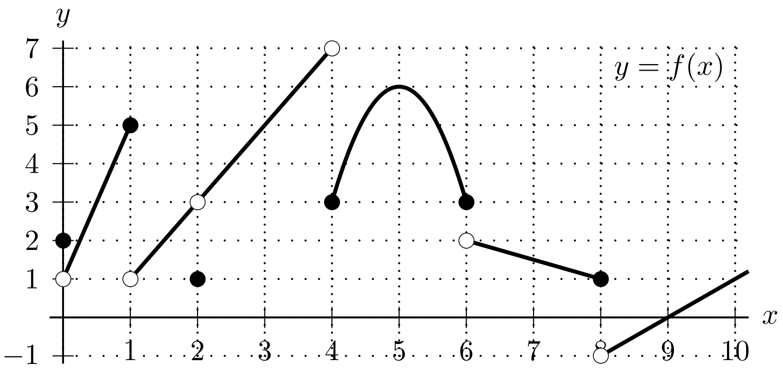
ii. The amount of caffeine that will result in Oren finishing his homework in 21 minutes is approximately 10 milligrams greater than the amount of caffeine that Oren will need in order to finish his homework in 20 minutes.

iii. The rate at which Oren is consuming caffeine 20 minutes into his homework assignment is decreasing by 10 milligrams per minute.

iv. [✓ In order to complete tonight's assignment in 19 rather than 20 minutes, Oren needs to consume about 10 milligrams of additional caffeine.]

v. If Oren consumes 20 milligrams of caffeine, then he will finish tonight's assignment approximately 10 minutes faster than if he consumes no caffeine.

6. [12 points] A portion of the graph of a function f is shown below.



a. [2 points] Find an equation for the tangent line to the graph of $y = f(x)$ at $x = 0.5$

Answer: $y =$ $4x + 1$

For parts **b-d** below, evaluate the given expression. If the expression does not represent a real number, write DNE.

b. [2 points] $\lim_{u \to 2} f(u)$

Answer: $\lim_{u \to 2} f(u) =$ 3

c. [2 points] $f'(f(7))$

Answer: $f'(f(7)) =$ $f'(1.5) = 2$

d. [2 points] $\ln(f'(9))$

Answer: $\ln(f'(9)) =$ $\ln(1) = 0$

For each of the following statements, find all real numbers c in the interval $0 \le c \le 10$ such that the statement holds. If there are no such values of c, write NONE.

e. [2 points] $\lim_{x \to c^+} f(x) = f(c)$ and f is not continuous at c.

Answer: $c = 4$

f. [2 points] $f(c)f'(c) = 0$.

Answer: $c = 5, 9$

7. [11 points] The players on the U-M football team rehydrate during the 20 minute halftime break. Suppose that during the first game of the 2014 season, researchers in the athletic department tracked the team's consumption of sports drink during halftime. Every time another 6 gallons of sports drink was consumed by the players, the time was recorded. Some of the data is provided below.

total amount of sports drink consumed by team (in gallons)	0	6	12	18	24
time since the start of halftime (in minutes)	0	0.6	2.4	5.2	10.0

Let $G(t)$ be the total number of gallons of sports drink the team consumed during the first t minutes of halftime. Assume that $G(t)$ is continuous and differentiable.

a. [3 points] Recall that halftime is 20 minutes long. Suppose that the average rate at which the football team consumed sports drink during the last 10 minutes of halftime is 0.7 gallons per minute. Find $G(20)$. *Remember to show your work clearly.*

Solution: The team consumed

$$(0.7 \text{ gallons/minute})(10 \text{ minutes}) = 7 \text{ gallons}$$

during the last 10 minutes of halftime. Thus, $G(20) = G(10) + 7 = 24 + 7 = 31$.

Answer: $G(20) =$ 31

b. [2 points] Which of the following statements is best supported by the data in the table? *Circle the* ONE *best answer.*

i. $G'(t)$ is an increasing function.

ii. [✓ $G'(t)$ is a decreasing function.]

iii. $G'(t)$ is a constant function.

c. [3 points] Approximate the instantaneous rate at which the football team was consuming sports drink 8 minutes after the start of halftime.
Include units, and remember to show your work clearly.

Solution: This is given by

$$G'(8) \approx \frac{G(10) - G(5.2)}{10 - 5.2} = \frac{24 - 18}{10 - 5.2} = 1.25 \text{ gallons/minute}$$

Answer: 1.25 **gallons/minute**

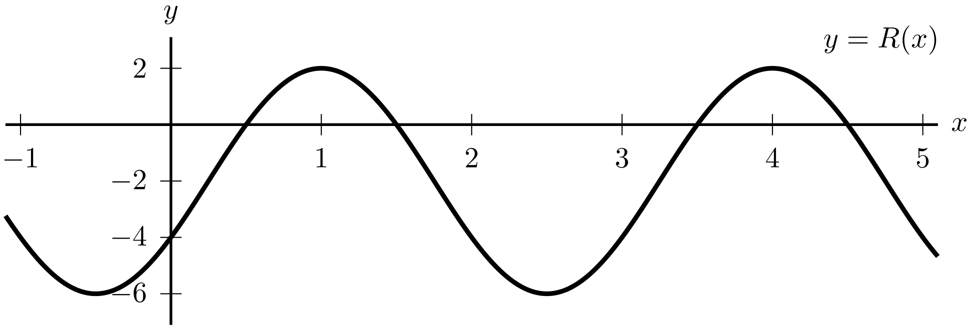
d. [3 points] Assume that $G(t)$ is invertible and that G^{-1} is differentiable. Approximate $(G^{-1})'(3)$. *Remember to show your work clearly.*

Solution:

$$(G^{-1})'(3) \approx \frac{G^{-1}(6) - G^{-1}(0)}{6 - 0} = \frac{0.6 - 0}{6 - 0} = 0.1 \text{ minutes/gallon}$$

Answer: $(G^{-1})'(3) \approx$ 0.1 **minutes/gallon**

8. [6 points] Given below is the graph of a sinusoidal function $R(x)$.



Find a possible formula for $R(x)$.

Solution: The graph shown above is of a sinusoidal function with amplitude 4, period 3, and midline $y = -2$. We first consider the graph of

$$y = 4\cos\left(\frac{2\pi}{3}x\right) - 2\,.$$

This graph has the proper amplitude, period, and midline. We shift this graph over to the right 1 unit to obtain the graph of $y = R(x)$. Thus, one possible formula for $R(x)$ is given by

$$R(x) = 4\cos\left(\frac{2\pi}{3}(x-1)\right) - 2.$$

Answer: $R(x) =$ $4\cos\left(\frac{2\pi}{3}(x-1)\right) - 2$

9. [4 points] The table below gives several values of a function $w(x)$.

x	4.5	4.9	4.99	5	5.01	5.1	5.5
$w(x)$	-0.879	-0.154	-0.015	0	0.060	0.630	3.750

Use the information in the table above to estimate the following limit.

$$\lim_{h\to 0^-} \frac{w(5+h)}{h}$$

Clearly show any computations that you use to make this estimate.

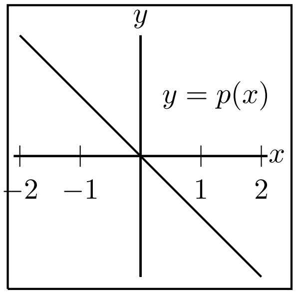
Solution: The left limit can be approximated by $w(5+h)/h$ for small negative values of h. The table of values provided for w allows us to compute this when $h = -0.1$ and when $h = -0.01$. The results are shown in the table below.

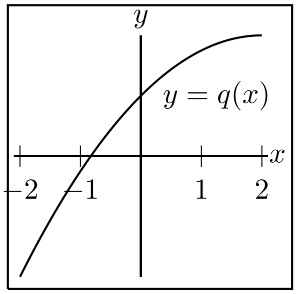
h	$\frac{w(5+h)}{h}$
-0.1	$\frac{w(4.9)}{-0.1} = 1.54$
-0.01	$\frac{w(4.99)}{-.01} = 1.5$

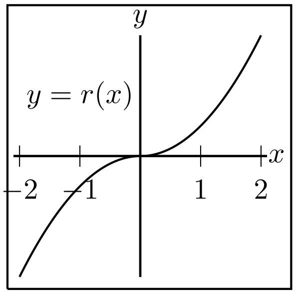
Using these values, we estimate that the desired left-hand limit is approximately 1.5.

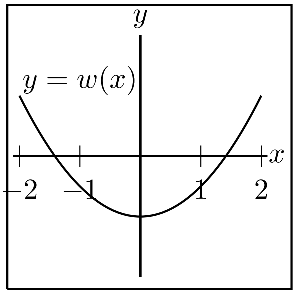
Answer: $\lim_{h\to 0^-} \frac{w(5+h)}{h} \approx$ 1.5

10. [8 points] The graphs of four differentiable functions p, q, r, and w are shown below.





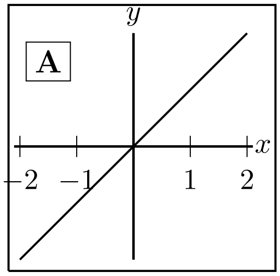


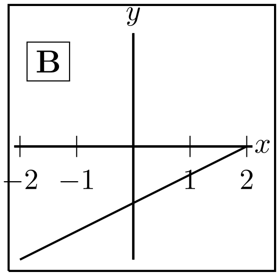


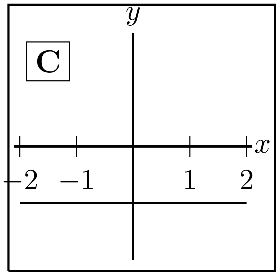
Graph of p': C **Graph of q':** H **Graph of r':** F **Graph of w':** A

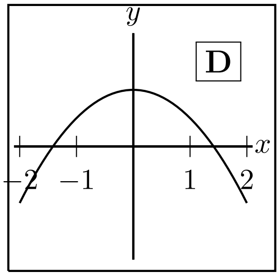
For each function above, choose the ONE graph from the choices A-L below that best indicates the behavior of the derivative of that function. Write the capital letter of your choice on the provided answer blank. You may use a letter more than once if necessary. Any unclear answers will be marked as incorrect. (Note that the scale on the y-axis is not indicated on any of the graphs and may vary between the graphs.)

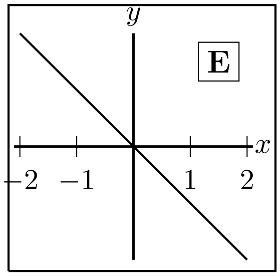
Answer Choices:

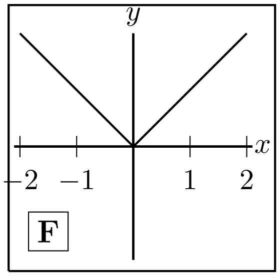


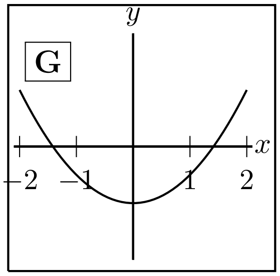


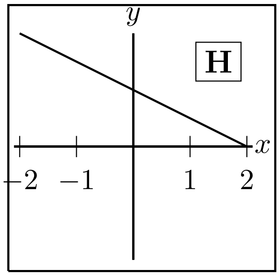


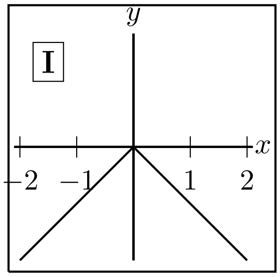


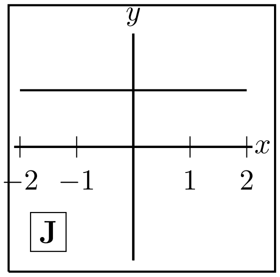


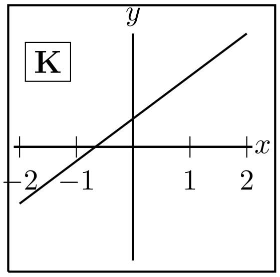


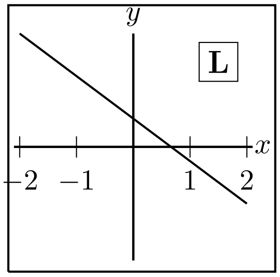




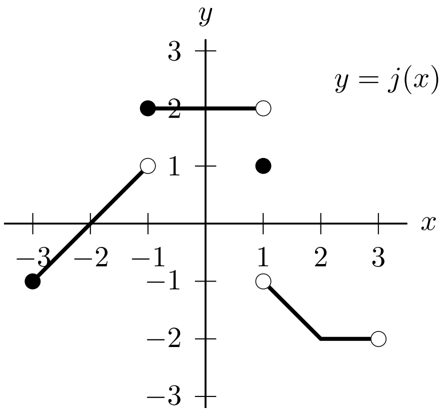




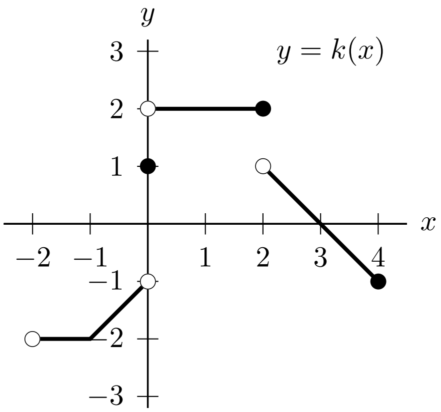


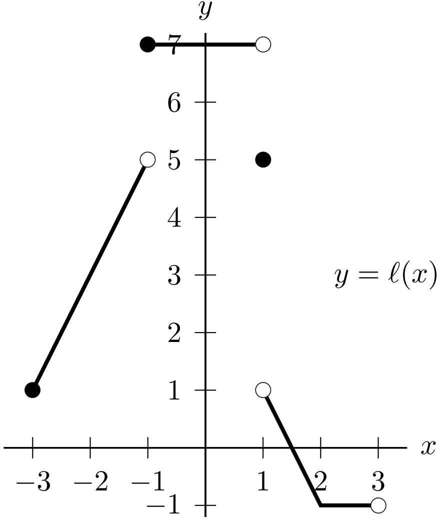


11. [6 points] Below is the graph of a function $j(x)$.



The graphs below show two other functions $k(x)$ and $\ell(x)$ which are transformations of $j(x)$. Write a formula for each in terms of j and x.





Answer: $k(x) =$ $j(-(x-1))$ and $\ell(x) =$ $2j(x) + 3$

12. [3 points] Find a formula for one polynomial $p(x)$ that satisfies both of the following properties.

- The degree of $p(x)$ is at least 5.
- The domain of the function $\ln(p(x))$ is the interval $(-\infty, \infty)$.

Note that this problem may have more than one correct answer. You only need to find one correct answer.

Solution: Since the domain of $\ln(x)$ is the interval $(0, \infty)$, a number x is in the domain of $\ln(p(x))$ if and only if $p(x) > 0$. So we need to find a polynomial of degree at least 5 that is always positive. Note that any such polynomial has to have even degree (since the end behavior of an odd degree polynomial differs on the two ends). One possible answer is $p(x) = x^6 + 1$.

Answer: $p(x) =$ $x^6 + 1$