Math 115 — Final Exam

April 19, 2012

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 10 pages including this cover. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	8	
2	15	
3	16	
4	12	
5	12	
6	12	
7	12	
8	13	
Total	100	

1. [8 points] A ship is sailing out to sea from a dock, moving in a straight line perpendicular to the coast. At the same time, a person is running along the coast toward the dock, hoping desperately to jump aboard the departing ship. Let $b(t)$ denote the distance in feet between the ship and the dock t seconds after its departure, and let $p(t)$ denote the distance in feet between the person and the dock t seconds after the ship's departure. The situation is depicted below for your reference:

Suppose that 10 seconds after the ship's departure, it is 40 feet from the dock and is sailing away at a speed of 20 ft/sec. At the same moment, the person is 30 feet from the dock and running toward it at 14 ft/sec.

a. [2 points] What is $b'(10)$? What is $p'(10)$?

Solution: Since the distance between the ship and the dock is increasing at 20 ft/sec while the distance between the person and the dock is decreasing at 14 ft/sec, we have

$$b'(10) = 20, \text{ and } p'(10) = -14.$$

b. [6 points] Is the distance between the person and the ship increasing or decreasing 10 seconds after the ship's departure? How fast is it increasing or decreasing? (Include units in your answer, and keep in mind that distance is measured along a straight line joining the person and the ship.)

Solution: If the distance between the ship and the person is denoted t seconds after the ship's departure is denoted $d(t)$, then by the Pythagorean Theorem,

$$b(t)^2 + p(t)^2 = d(t)^2.$$

Differentiating this gives:

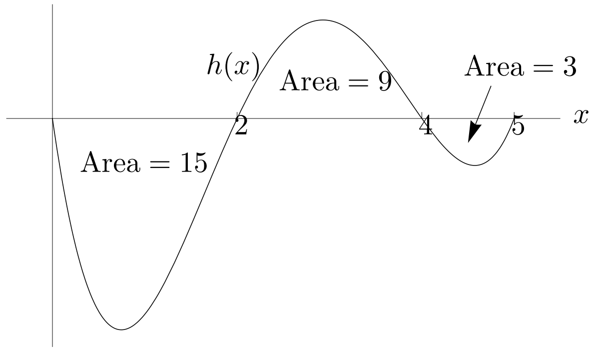
$$2b(t)b'(t) + 2p(t)p'(t) = 2d(t)d'(t).$$

We know that $b(10) = 40$, $p(10) = 30$, and $d(10) = \sqrt{40^2 + 30^2} = 50$. If we plug this information as well as the numbers from part (a) into the above equation and solve for $d'(t)$, we find

$$d'(t) = \frac{760}{100} = 7.6.$$

Since this is positive, the distance is increasing at 7.6 ft/sec.

2. [15 points] Using the graph of $h(x)$ shown below, compute each of the following quantities. If there is not enough information to compute the given quantity, write "not enough information". You do not need to explain your answers.



a. [3 points] $\displaystyle\int_2^0 \Big(h(x)+2\Big)dx$

Solution:

$$\int_2^0 \Big(h(x)+2\Big)dx = -\int_0^2 h(x)dx - \int_0^2 2dx = 15-4 = 11.$$

b. [2 points] $\displaystyle\int_0^5 3h(y)dy$

Solution:

$$\int_0^5 3h(y)dy = 3(-15+9-3) = -27.$$

c. [3 points] $\displaystyle\int_8^9 h(x-4)dx$

Solution:

$$\int_8^9 h(x-4)dx = \int_4^5 h(x)dx = -3.$$

d. [3 points] The average value of $h(x)$ on the interval $[-2,2]$, assuming that $h(x)$ is an even function.

Solution:

$$\frac{1}{4}\int_{-2}^2 h(x)dx = \frac{1}{4}\cdot 2\cdot \int_0^2 h(x)dx = \frac{-15}{2}.$$

e. [2 points] $H(2)$, where H is an antiderivative of h

Solution: Not enough information.

f. [2 points] $H(2)-H(0)$, where H is an antiderivative of h

Solution: By the fundamental theorem of calculus,

$$H(2)-H(0) = \int_0^2 h(x)dx = -15.$$

3. [16 points] A truck is driving along a straight highway. The function $v(t)$ gives its velocity in m/sec after t seconds on the highway, and the function $a(t)$ gives its acceleration in m/sec^2 after t seconds on the highway. Consider the following table of values for $v(t)$ and $a(t)$, keeping in mind that $a(t) = v'(t)$.

t	0	10	20	30	40
$v(t)$	13	21	27	32	35
$a(t)$	1.0	0.7	0.55	0.4	0.2

In the following questions, **include units** wherever appropriate.

a. [3 points] Use a tangent line approximation to estimate the velocity of the truck after 33 seconds on the highway.

Solution:

$$v(33) \approx v(30) + 3 \cdot v'(30) = 32 + 3 \cdot 0.4 = 33.2 \text{ m/sec.}$$

b. [3 points] Do you expect that your approximation in part (a) is an overestimate or an underestimate? Briefly explain your answer based on the information in the table.

Solution: Since $a(t) = v'(t)$ is decreasing, $v(t)$ is concave down, so the approximation is an overestimate.

c. [4 points] Use a left-hand sum with four subdivisions to approximate the distance traveled by the truck in the first 40 seconds on the highway. Write out each term of the sum as well as the final answer.

Solution:

$$10(13 + 21 + 27 + 32) = 930 \text{ m.}$$

d. [3 points] Do you expect that your approximation in part (c) is an overestimate or an underestimate? Briefly explain your answer based on the information in the table.

Solution: Since $v(t)$ is increasing, the approximation is an underestimate.

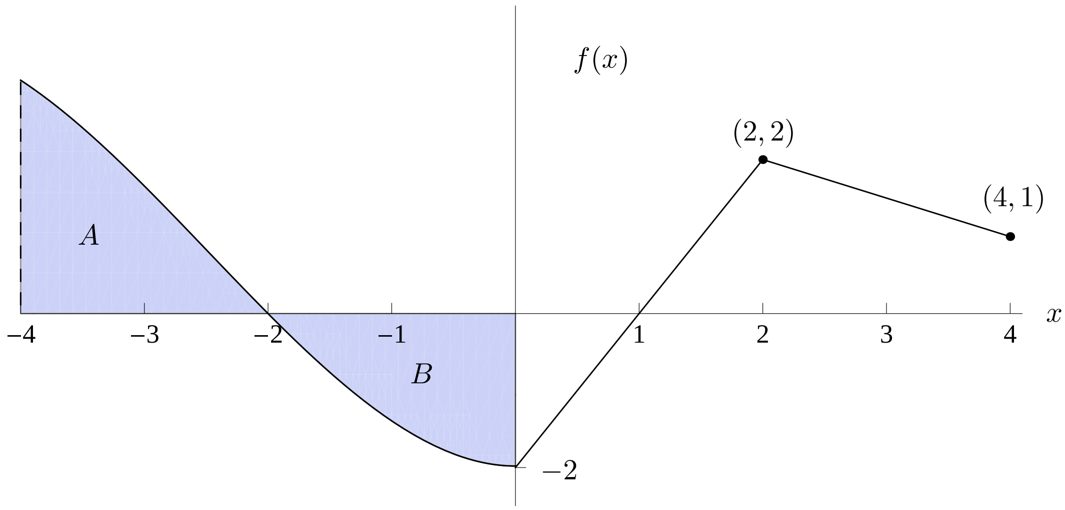
e. [3 points] How frequently would velocity measurements need to be made in order to ensure that the left-hand sum and right-hand sum approximating the distance traveled in the first 40 seconds agree to within 1 meter?

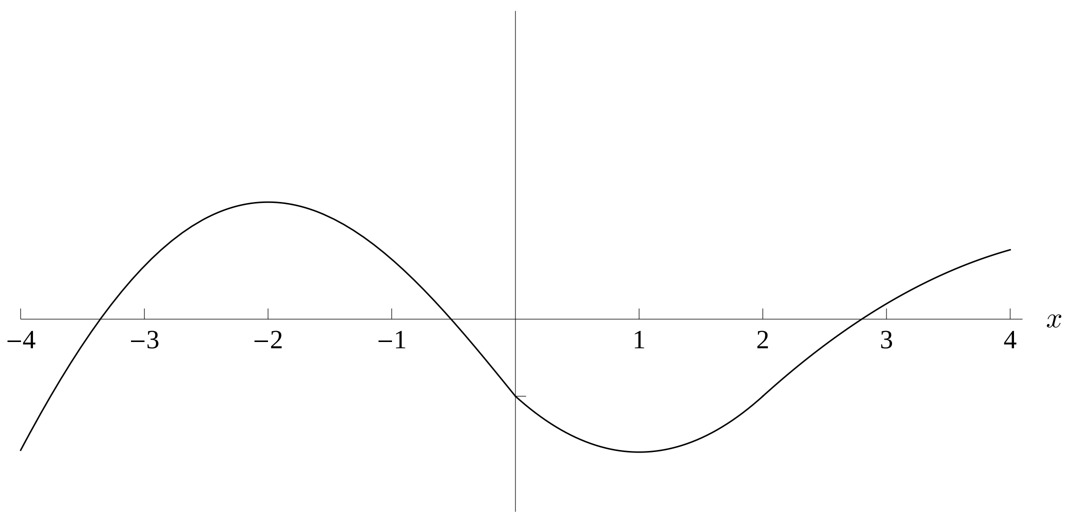
Solution: We need

$$1 = (35 - 13)\Delta t,$$

so $\Delta t = 1/22$, or in other words, measurements should be made 22 times per second.

4. [12 points] The graph of a function $f(x)$ is shown below. The area of shaded region A is 3.1 and the area of shaded region B is 2.2. On the axes provided, sketch a well-labeled graph of an antiderivative $F(x)$ of $f(x)$ satisfying $F(0) = -1$. Indicate the x- **and** y- coordinates of all critical points and inflection points of $F(x)$ in the space provided.





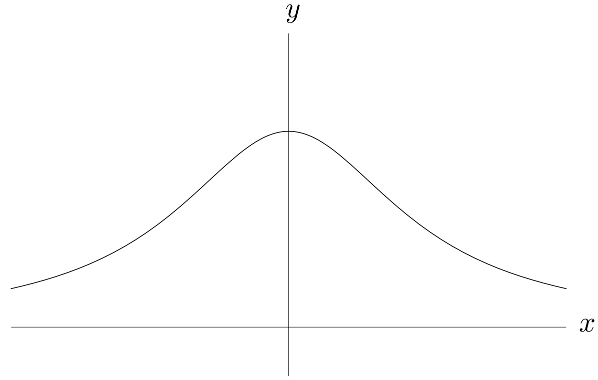
Coordinates of critical points: $\underline{(-2, 1.2), (1, -2)}$

Coordinates of inflection points: $\underline{(0, -1), (2, -1)}$

5. [12 points] A rectangle has one side on the x-axis and two vertices on the curve

$$y = \frac{36}{9+x^2}.$$

This curve is graphed below. Find the x- and y- coordinates of all four vertices of the rectangle with largest area. You must show that your vertices maximize the area of the rectangle.



Solution: The area of such a rectangle, if its lower-right corner is at $(x, 0)$, is

$$A = 2x\left(\frac{36}{9+x^2}\right) = \frac{72x}{9+x^2}.$$

Differentiating this gives

$$\frac{dA}{dx} = \frac{(9+x^2)72 - 72x(2x)}{(9+x^2)^2} = \frac{648 - 72x^2}{(9+x^2)^2},$$

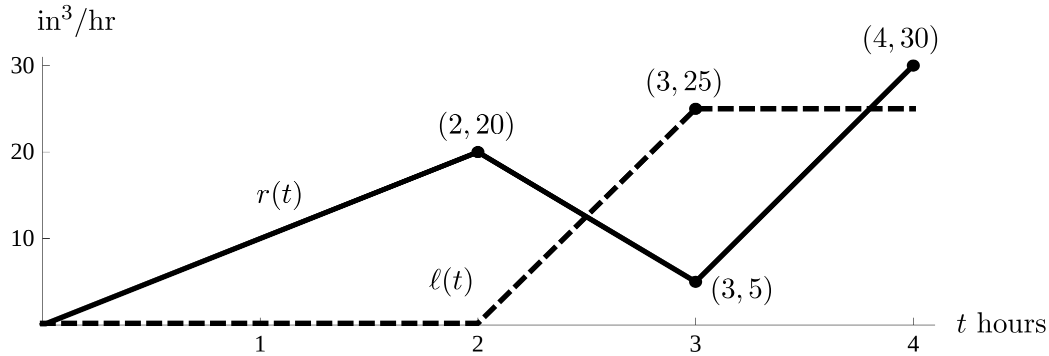
so there is a critical point where the numerator is zero, which is

$$x = 3.$$

We can verify using the first derivative test that this is a local maximum (for example, by noticing that dA/dx is positive at $x = 0$ and negative at $x = 4$), and since it is the only critical point on the domain $(0, \infty)$ we are considering, it must be the global maximum. Plugging into the equation for the curve shows that $y = 2$ when $x = 3$. Therefore, the four vertices of the rectangle are

$$(\pm 3, 2), (\pm 3, 0).$$

6. [12 points] A large bucket is left outside during a storm, and the bucket begins to fill with rain. The rain starts at midnight, at which point the bucket is empty. At 2am, the bucket springs a leak and some water begins to drip out of it. The function $r(t)$ is the rate at which rain is falling into the bucket t hours after midnight, measured in in^3/hr, while the function $\ell(t)$ is the rate at which water is leaking out of the bucket t hours after midnight, measured in in^3/hr. These functions are graphed below.



Be sure to **include units** in your answers to the following questions. No explanation is necessary, but partial credit may be given for correct work. Assume the bucket is big enough that it never overflows during the storm.

a. [3 points] How much water was in the bucket at 3am?

Solution: The amount of water that has entered is the area under the solid curve from 0 to 3, which is 32.5in^3, and the amount that has exited is the area under the dashed curve from 0 to 3, which is 12.5in^3. Thus, there is 20in^3 of water in the bucket at 3am.

b. [2 points] At what time was the amount of water in the bucket greatest?

Solution: This occurs when $r(t) = \ell(t)$ and $r(t) - \ell(t)$ is going from positive to negative, which is at $t = 2.5$, or 2:30am.

c. [3 points] What is the largest amount of water that was in the bucket between midnight and 4am?

Solution: This is the amount that was in the bucket at 2:30am, which is $\int_0^{2.5}(r(t) - \ell(t))dt = 25\text{in}^3$.

d. [2 points] At what time was the amount of water in the bucket increasing fastest?

Solution: This is when $r(t) - \ell(t)$ is largest, which is at 2am.

e. [2 points] Write an integral expressing the average rate at which rain fell into the bucket over the period from midnight to 4am. You do not need to evaluate your integral.

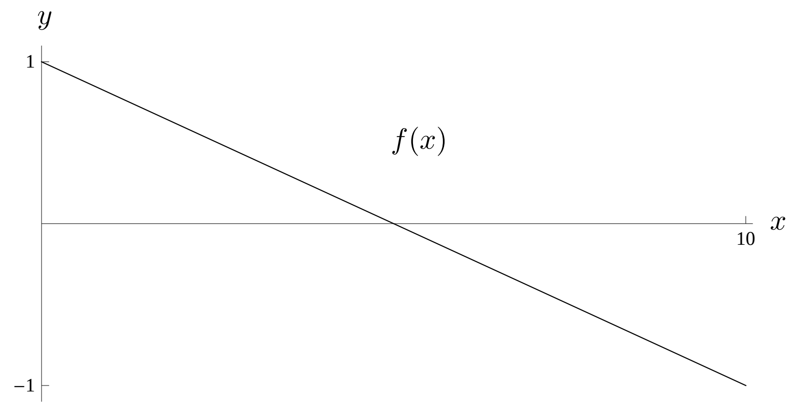
Solution:

$$\frac{1}{4}\int_0^4 r(t)dt$$

7. [12 points] In each of the following questions, sketch a well-labeled graph that satisfies the given conditions. You do not need to explain your answer, but you should label any relevant features of your graph. Each part of this problem is independent of the others and there may be many correct solutions.

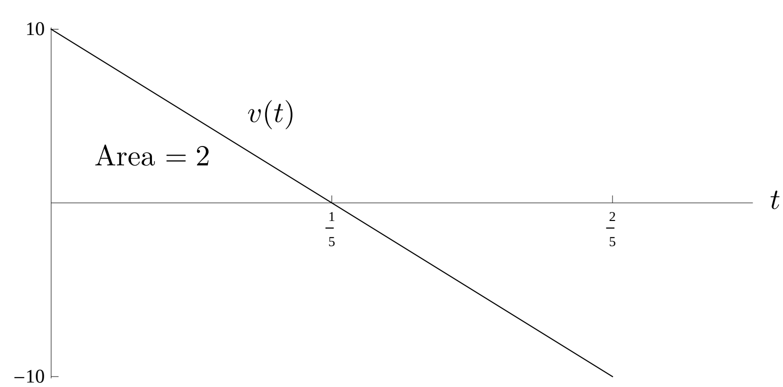
a. [3 points] Sketch the graph of a function $f(x)$ so that $\int_0^{10} f(x)dx = 0$ and $f(0) > 0$.

Solution:



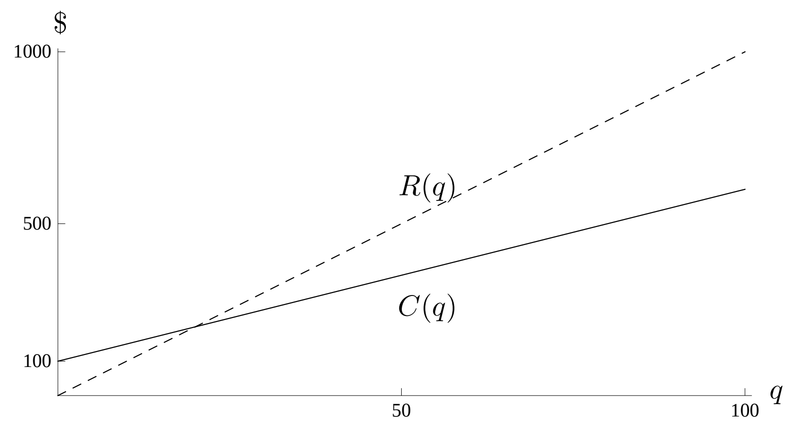
b. [3 points] Sketch the graph of a function $v(t)$ giving the velocity in ft/sec of a bird at time t seconds, assuming it leaps upward from the ground at an initial velocity of 10 ft/sec at time $t = 0$ and hovers momentarily at a height of 2 feet before falling back to the ground. Positive velocities should indicate upward motion of the bird, and you should label the aspects of your graph that correspond to each of the given quantities.

Solution:



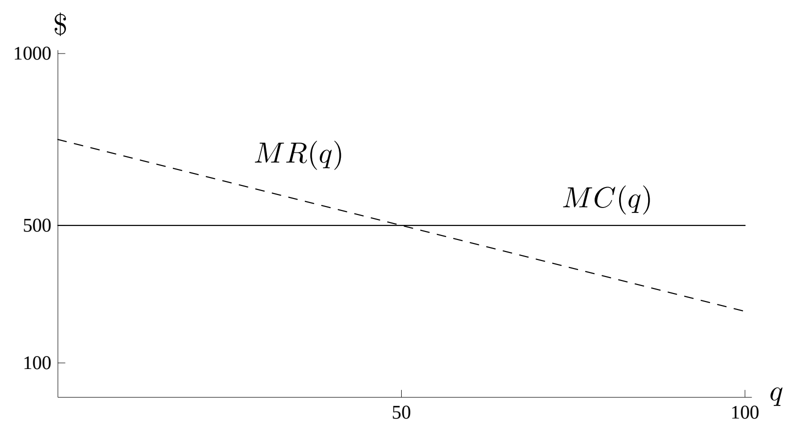
c. [3 points] On a single set of axes, sketch the graphs of functions $C(q)$ giving the cost of producing q units of a good and the revenue $R(q)$ obtained by selling q units. Assume that fixed costs are \$100 and the item is sold for \$10 per unit.

Solution:



d. [3 points] On a single set of axes, sketch the graphs of marginal cost $MC(q)$ and marginal revenue $MR(q)$ functions so that profit is maximized at $q = 50$ units.

Solution:



8. [13 points] Use the family of functions of the form

$$f(x) = ax - \ln(1 + e^{bx})$$

to answer the following questions. The constants a and b are both positive.

a. [4 points] Use the given formula for $f(x)$ to give an explicit expression for the limit definition of $f'(x)$. Check your expression carefully, as **no partial credit** will be given on this part of the problem. Do not evaluate your expression.

Solution:

$$f'(x) = \lim_{h\to 0} \frac{a(x+h) - \ln(1 + e^{b(x+h)}) - ax + \ln(1 + e^{bx})}{h}.$$

b. [4 points] Compute $f'(x)$ using the rules of differentiation. Do not try to evaluate your expression from (a).

Solution:

$$f'(x) = a - \frac{be^{bx}}{1 + e^{bx}}.$$

c. [5 points] When $a < b$, the function $f(x)$ has a critical point at

$$x = \frac{1}{b} \ln\left(\frac{a}{b-a}\right).$$

Using the **second-derivative test**, determine whether this critical point is a local maximum, local minimum, or neither.

Solution: The second derivative is

$$f''(x) = -\frac{(1 + e^{bx})b^2 e^{bx} - be^{bx}be^{bx}}{(1 + e^{bx})^2} = -\frac{b^2 e^{bx}}{(1 + e^{bx})^2}.$$

This is always negative, so the critical point is a local maximum.