Math 115 — Final Exam

April 28, 2014

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 11 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
8. Include units in your answer where that is appropriate.
9. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones.
10. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	11	
2	10	
3	11	
4	13	
5	9	
6	8	
7	7	
8	11	
9	9	
10	11	
Total	100	

1. [11 points] The table below gives several values of a function $f(x)$ and its derivative. Assume that both $f(x)$ and $f'(x)$ are defined and differentiable for all x.

x	0	1	2	3	4	5	6
$f(x)$	0	3	4	2	-1	-3	5
$f'(x)$	4	2	-1	-5	-2	7	9
$f''(x)$	-1	-3	-5	0	4	3	1

Compute each of the following. Do not give approximations. If it is not possible to find the value exactly, write NOT POSSIBLE

a. [2 points] Find $\int_0^4 f''(x)\,dx$.

Solution: By the Fundamental Theorem of Calculus,

$$\int_0^4 f''(x)\,dx = f'(4) - f'(0) = -2 - 4 = -6.$$

Answer: $\int_0^4 f''(x)\,dx =$ -6

b. [2 points] Find $\int_2^5 (3f(x)+1)\,dx$.

Solution: In order to evaluate this exactly, we would need to know an antiderivative of $f(x)$. Since we don't know one, this is not possible to evaluate exactly.

Answer: $\int_2^5 (3f(x)+1)\,dx =$ NOT POSSIBLE

c. [3 points] Find the average value of $4f'(x)+x$ on the interval $[1,6]$.

Solution: The average value can be computed as an integral. Since an antiderivative of $4f'(x)+x$ is $4f(x)+\frac{1}{2}x^2$, we can compute the exact value of this integral with the Fundamental Theorem of Calculus:

$$\frac{1}{6-1}\int_1^6 \left(4f'(x)+x\right)\,dx = \frac{1}{5}\left(\left(4f(6)+\frac{1}{2}6^2\right)-\left(4f(1)+\frac{1}{2}1^2\right)\right) = 5.1$$

Answer: 5.1

d. [4 points] Assuming that $f(x)$ is an odd function, find $\int_{-3}^3 f(x)\,dx$ and $\int_{-3}^3 f'(x)\,dx$.

Solution: Note that

$$\int_{-3}^3 f(x)\,dx = \int_{-3}^0 f(x)\,dx + \int_0^3 f(x)\,dx,$$

and since $f(x)$ is an odd function, the two integrals on the right cancel out, leaving us with 0.

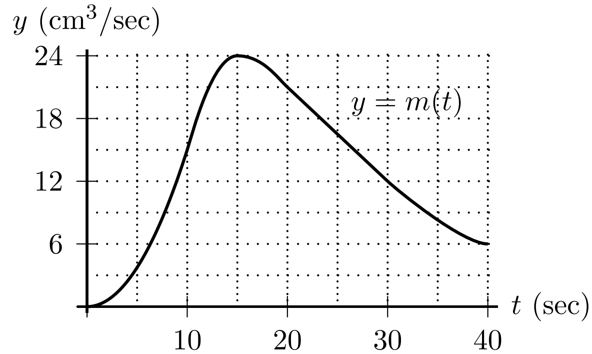
Also, since $f(x)$ is odd, we have $f(-3) = -f(3)$, and hence by the Fundamental Theorem of Calculus,

$$\int_{-3}^3 f'(x)\,dx = f(3) - f(-3) = f(3) - (-f(3)) = 2f(3) = 4.$$

Answer: $\int_{-3}^3 f(x)\,dx =$ 0 and $\int_{-3}^3 f'(x)\,dx =$ 4

2. [10 points]

Kathy puts a very large marshmallow in the microwave for forty seconds and watches as it inflates. Let $m(t)$ be the rate of change of the volume of the marshmallow, in cm^3/sec, t seconds after Kathy puts it in the microwave. The graph of $y = m(t)$ is shown to the right.



a. [2 points] Write a definite integral equal to the total change in volume, in cm^3, of the marshmallow while in the microwave. (You do not need to evaluate the integral.)

Answer: $\int_0^{40} m(t)\,dt$

b. [3 points] Estimate your integral from part (a) using a right-hand sum with $\Delta t = 10$. Be sure to write out all of the terms in the sum.

Solution: A right-hand sum from 0 to 40 with $\Delta t = 10$ will involve the values at $t = 10$, 20, 30, and 40:

$$10m(10) + 10m(20) + 10m(30) + 10m(40) = 10(15 + 21 + 12 + 6) = 540.$$

Since $m(t)$ has units of cm^3/sec and t has units of sec, the integral has units of cm^3, which agrees with it being a change in volume.

Answer: 540 $\mathbf{cm^3}$

c. [5 points] Assume that throughout its time in the microwave, the marshmallow is a cylinder. After 30 seconds in the microwave, the marshmallow is a cylinder with radius 4.5 cm and height 11 cm. At that moment, the height is increasing at 0.08 cm/sec. How fast is the radius of the marshmallow increasing at that moment?
Recall that the volume V of a cylinder of radius r and height h is $V = \pi r^2 h$, and remember to include units.

Solution: Differentiating both sides of the volume equation with respect to t yields

$$\frac{dV}{dt} = 2\pi r h\frac{dr}{dt} + \pi r^2\frac{dh}{dt}.$$

We are told that at this moment, $r = 4.5$, $h = 11$, and $\frac{dh}{dt} = 0.08$. Further, since $m(t) = \frac{dV}{dt}$, we can read from the graph above that at $t = 30$, we have $\frac{dV}{dt} = 12$. Plugging these in, we have

$$12 = 99\pi\frac{dr}{dt} + 1.62\pi,$$

so solving for $\frac{dr}{dt}$ yields a rate of about 0.022 cm/sec.

Answer: 0.022 **cm/sec**

3. [11 points] For positive constants a and b, the potential energy of a particle is given by

$$U(x) = a\left(\frac{5b^2}{x^2} - \frac{3b}{x}\right).$$

Assume that the domain of $U(x)$ is the interval $(0, \infty)$.

a. [2 points] Find the asymptotes of $U(x)$. If there are none of a particular type, write NONE.

Solution: We can get a common denominator and write

$$U(x) = a\frac{5b^2 - 3bx}{x^2}.$$

We see that there is a vertical asymptote at $x = 0$, where the denominator is zero, and a horizontal asymptote at $U = 0$, since the degree of the denominator is greater than the degree of the numerator.

Answer: Vertical asymptote(s): $x = 0$ Horizontal asymptote(s): $U = 0$

b. [6 points] Find the x-coordinates of all local maxima and minima of $U(x)$ in the domain $(0, \infty)$. If there are none of a particular type, write NONE. You must use calculus to find and justify your answers. Be sure to provide enough evidence to justify your answers fully.

Solution: First we find critical points by looking at where $U'(x)$ is undefined or zero. We have

$$U'(x) = a\left(-\frac{10b^2}{x^3} + \frac{3b}{x^2}\right) = a\frac{3bx - 10b^2}{x^3}.$$

There are no points in the domain of $U(x)$ where $U'(x)$ is undefined, but $U'(x)$ has a zero where $3bx - 10b^2 = 0$, or $x = \frac{10b}{3}$.
To classify this critical point, we can use the First or Second Derivative Test. We will use the Second Derivative Test here, so we compute

$$U''(x) = a\left(\frac{30b^2}{x^4} - \frac{6b}{x^3}\right) = \frac{6ab}{x^4}(5b - x).$$

Since a, b, and x^4 are always positive and $5b - x$ is positive at $x = \frac{10b}{3}$, we see that $U''\left(\frac{10b}{3}\right) > 0$, and hence $x = \frac{10b}{3}$ is a local minimum.
There are no other critical poitns to consider, so there are no local maxima.

Answer: Local max(es) at $x =$ NONE Local min(s) at $x =$ $\frac{10b}{3}$

c. [3 points] Suppose $U(x)$ has an inflection point at $(6, -14)$. Find the values of a and b. *Show your work, but you do not need to verify that this point is an inflection point.*

Solution: We already found $U''(x) = \frac{6ab}{x^4}(5b - x)$, so we see that the only potential inflection point occurs at $x = 5b$, the only place in the domain of $U(x)$ where $U''(x)$ is zero or undefined. Hence $5b = 6$ or $b = 1.2$.
Plugging in $x = 6$, $b = 1.2$, and $U = -14$ into the original equation for $U(x)$ yields

$$-14 = a\left(\frac{1}{5} - \frac{3}{5}\right) = -\frac{2a}{5}$$

and hence $a = 35$.

Answer: $a =$ 35 and $b =$ 1.2

4. [13 points] One of the ways Captain Christina likes to relax in her retirement is to go for long walks around her neighborhood. She has noticed that early every Tuesday morning, a truck delivers butter to a local bakery famous for its cookie dough. Consider the following functions:

- Let $C(b)$ be the bakery's cost, in dollars, to buy b pounds of butter.
- Let $K(b)$ be the amount of cookie dough, in cups, the bakery makes from b pounds of butter.
- Let $u(t)$ be the instantaneous rate, in pounds per hour, at which butter is being unloaded t hours after 4 am.

Assume that C, K, and u are invertible and differentiable.

a. [2 points] Interpret $K(C^{-1}(10)) = 20$ in the context of this problem. Use a complete sentence and include units.

Solution: If the bakery spends \$10 on butter, then it can make 20 cups of cookie dough.

b. [3 points] Interpret $\int_5^{12} K'(b)\,db = 40$ in the context of this problem. Use a complete sentence and include units.

Solution: 12 pounds of butter makes 40 cups more cookie dough than 5 pounds of butter does.

c. [2 points] Give a single mathematical equality involving the <u>derivative</u> of C which supports the following claim:
It costs the bakery approximately \$0.70 less to buy 14.8 pounds of butter than to buy 15 pounds of butter.

Answer: $C'(15) = 3.5$

d. [3 points] Give a single mathematical equality which expresses the following claim:
The number of pounds of butter unloaded between 5 and 8 am is twice as many as the bakery needs to make 5000 cups of cookie dough.

Answer: $\int_1^4 u(t)\,dt = 2K^{-1}(5000)$

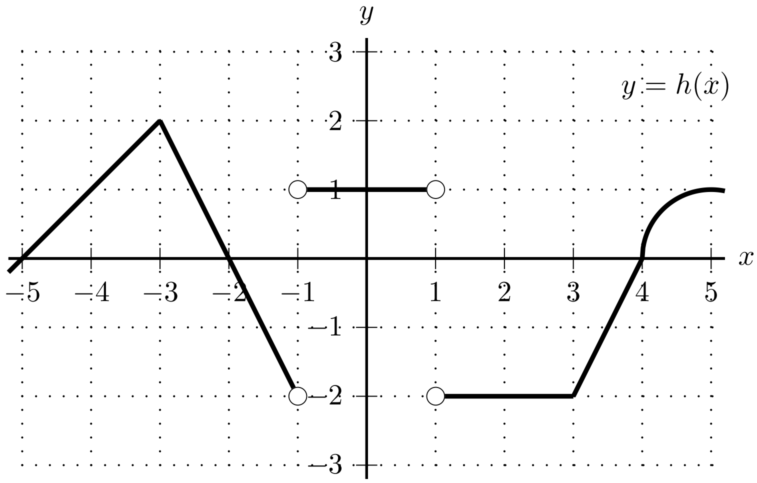
e. [3 points] Assume that $u(t) > 0$ and $u'(t) < 0$ for $0 \le t \le 4$ and that $u(2) = 800$.
Rank the following quantities in order from least to greatest by filling in the blanks below with the options I-IV.

I. 0 II. 800 III. $\int_1^2 u(t)\,dt$ IV. $\int_2^3 u(t)\,dt$

Solution: Since $u(t) > 0$, both integrals are greater than 0. Since $u'(t) < 0$, $u(t)$ is a decreasing function. Estimating $\int_1^2 u(t)\,dt$ with a right sum with one subdivision yields an underestimate of 800, and likewise, estimating $\int_2^3 u(t)\,dt$ with a left sum with one subdivision yields an overestimate of 800.

$0 < \int_2^3 u(t)\,dt < 800 < \int_1^2 u(t)\,dt$

5. [9 points] The graph of a portion of $y = h(x)$ is shown below.



Note: The portion of the graph of $h(x)$ between $x = 4$ and $x = 5$ is part of a circle of radius 1 centered at the point $(5, 0)$.

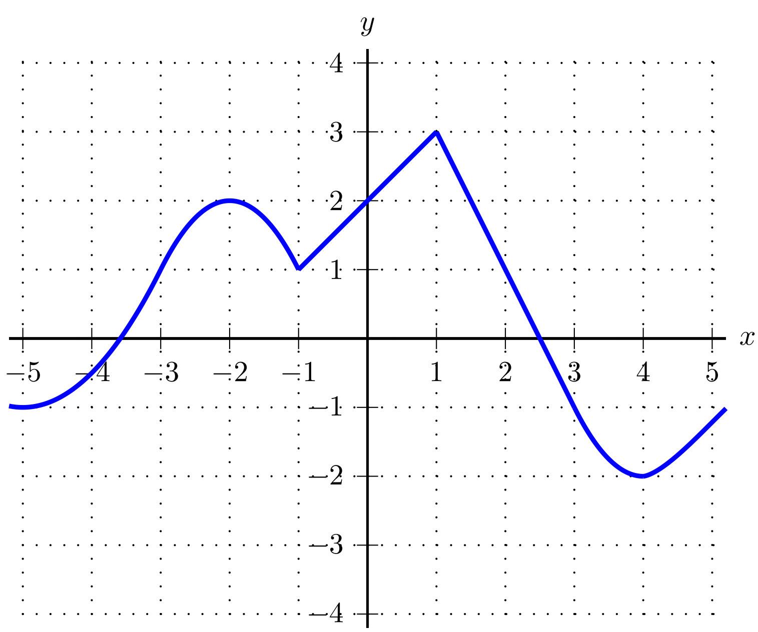
Let $H(x)$ be the continuous antiderivative of $h(x)$ with $H(0) = 2$.

a. Complete the following table with the exact values of $H(x)$.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$H(x)$	-1	-0.5	1	2	1	2	3	1	-1	-2	$-2+\frac{\pi}{4}$

b. On the axes below, sketch the graph of $y = H(x)$. Be sure that you pay close attention to each of the following:

- where $H(x)$ is and is not differentiable
- the values of $H(x)$ from the table above
- the sign of $H(x)$, where $H(x)$ is increasing/decreasing/constant, and the concavity of $H(x)$



6. [8 points] Suppose that a ring made entirely of gold and platinum is made from g ounces of gold and p ounces of platinum and that gold costs h dollars per ounce and platinum costs k dollars per ounce. Then the value, in dollars, of the ring is given by

$$v = gh + pk.$$

a. [3 points] Pat has a ring made entirely of gold and platinum. Pat's ring is made from 0.25 ounces of gold and 0.15 ounces of platinum. Suppose that the cost of gold is decreasing at an instantaneous rate of \$20 per ounce per year while the cost of platinum is increasing at an instantaneous rate of \$30 per ounce per year. At what instantaneous rate is the value of Pat's ring increasing or decreasing? *Remember to include units in your answer.*

Solution: In this setting, g is constant at 0.25 and p is constant at 0.15, and both h and k are changing. Differentiating with respect to t, we have

$$\frac{dv}{dt} = 0.25\frac{dh}{dt} + 0.15\frac{dk}{dt}.$$

Plugging in $\frac{dh}{dt} = -20$ and $\frac{dk}{dt} = 30$ yields $\frac{dv}{dt} = -5 + 4.5 = -0.5$.

Answer: The value of Pat's ring is (circle one) INCREASING $\boxed{\text{DECREASING}}$

at a rate of \$0.50 **per ounce per year**

b. [5 points] Jordan wants to design a ring made entirely of gold and platinum with a current value of \$900. Currently, gold costs \$1200 per ounce and platinum costs \$1500 per ounce. Let $w(p)$ be the total weight of Jordan's ring, in ounces, if p ounces of platinum are used.

(i) In the context of this problem, what is the domain of $w(p)$?

Solution: If the ring is all gold, then we use 0 ounces of platinum. Since the ring is worth \$900, the most platinum we could possibly use is $900/1500 = 0.6$ ounces.

Answer: $[0, 0.6]$

(ii) Find a formula for $w(p)$. No variables other than p should appear in your answer.

Solution: Since the value must be \$900, we have $900 = 1200g + 1500p$, or $g = 0.75 - 1.25p$. The total weight is therefore $w(p) = g + p = 0.75 - 0.25p$.

Answer: $w(p) =$ $0.75 - 0.25p$

(iii) How much gold and platinum should be in the ring if Jordan wants to minimize the weight of the ring? *You do not need to justify your answer.*

Solution: Since $w(p)$ is linear with negative slope, the smallest value will occur when p is greatest. Therefore, it occurs at $p = 0.6$, the right endpoint of our domain, where we use 0.6 ounces of platinum and 0 ounces of gold.

Answer: 0 ounces of gold and 0.6 ounces of platinum

7. [7 points] The record time for the 100 meter dash is 9.58 seconds, set by Usain Bolt at a race in 2009. Let $v(t)$ be Bolt's velocity, in meters per second, t seconds after Bolt starts the race. Several values of $v(t)$ are shown below. Assume that $v(t)$ is an increasing function for the first three seconds of the race.

t	0	0.5	1	1.5	2	2.5	3
$v(t)$	0.6	3.5	5.8	7.7	9.1	10.1	10.6

(Numbers are based on data collected at the 12th IAAF World Championships in Athletics.[1]*)*

a. [2 points] Estimate Bolt's instantaneous acceleration 1.75 seconds after Bolt starts the race. *Remember to include units.*

Solution: Since acceleration is the rate of change of velocity, we estimate this with a difference quotient between $t = 1.5$ and $t = 2$:

$$a(1.75) \approx \frac{9.1 - 7.7}{2 - 1.5} = 2.8.$$

Answer: 2.8 $\mathbf{m/sec^2}$

b. [3 points] Based on the data provided, give the best possible underestimate of the distance run by Bolt during the first 3 seconds of his race. *Be sure to show your work, and remember to include units.*

Solution: Since $v(t)$ is an increasing function, left sums provide underestimates. The best possible underestimate given the data we have, then, is a left sum with $\Delta t = 0.5$:

$$\begin{aligned} &0.5v(0) + 0.5v(0.5) + 0.5v(1) + 0.5v(1.5) + 0.5v(2) + 0.5v(2.5) \\ &\qquad = 0.5(0.6 + 3.5 + 5.8 + 7.7 + 9.1 + 10.1) \\ &\qquad = 18.4 \end{aligned}$$

Answer: 18.4 **m**

c. [2 points] How often would we need to measure Bolt's velocity so that the difference between the best possible underestimate and the best possible overestimate of the distance he runs in the first 3 seconds is 2 meters? *Be sure to show your work.*

Solution: If the velocity is measured in equally spaced intervals of width Δt, the difference between the left sum and right sum approximations is

$$(v(3) - v(0))\Delta t = (10.6 - 0.6)\Delta t = 10\Delta t.$$

For this to equal 2, then, we need $\Delta t = 0.2$.

Answer: Velocity must be measured every 0.2 seconds.

[1] Data available at `http://berlin.iaaf.org`

8. [11 points] A function $g(x)$ and its derivative are given by

$$g(t) = 10e^{-0.5t}(t^2 - 2t + 2) \qquad \text{and} \qquad g'(t) = -10e^{-0.5t}(0.5t^2 - 3t + 3).$$

a. [2 points] Find the t-coordinates of all critical points of $g(t)$. If there are none, write NONE. For full credit, you must find the exact t-coordinates.

Solution: Since $g'(t)$ is defined for all t, the only critical points occur where $g'(t) = 0$. To find these t values, we use the Quadratic Formula:

$$t = 3 \pm \sqrt{9-6} = 3 \pm \sqrt{3}.$$

Answer: Critical point(s) at $t =$ $3-\sqrt{3}$, $3+\sqrt{3}$

b. [6 points] For each of the following, find the values of t that maximize and minimize $g(t)$ on the given interval. Be sure to show enough evidence that the points you find are indeed global extrema. For each answer blank, write NONE in the answer blank if appropriate.

(i) Find the values of t that maximize and minimize $g(t)$ on the interval $[0, 8]$.

Solution: Since $[0, 8]$ is a closed interval, by the Extreme Value Theorem, $g(t)$ must have both a global max and a global min, occuring either at a critical point or at an endpoint. We therefore make a table of values at $t = 0, t = 3-\sqrt{3} \approx 1.27$, $t = 3+\sqrt{3} \approx 4.73$, and $t = 8$:

t	0	1.27	4.73	8
$g(t)$	20	5.69	14.01	9.16

From the table, we see the global max at $t = 0$ and the global min at $t = 3-\sqrt{3} \approx 1.27$.

Answer: Global max(es) at $t =$ 0 Global min(s) at $t =$ $3-\sqrt{3}$

(ii) Find the values of t that maximize and minimize $g(t)$ on the interval $[4, \infty)$.

Solution: Note that only one of our critical points, $t = 3+\sqrt{3} \approx 4.73$, lies in this interval. Global extrema, if they exist, then, can only occur at $t = 4$ and $t = 3+\sqrt{3}$, so we make a table of these values:

t	4	4.73
$g(t)$	13.53	14.01

We must also consider the behavior as $t \to \infty$, the open endpoint of our interval. Note that $g'(t) < 0$ for $t > 3+\sqrt{3}$, so $g(t)$ is decreasing for $t > 3+\sqrt{3}$. So the global max occurs at the largest value in the table, at $t = 3+\sqrt{3} \approx 4.73$. As t gets larger and larger, $g(t) = \frac{10(t^2-2t+2)}{e^{0.5t}}$ tends to 0, as $e^{0.5t}$ grows faster than any polynomial in the long run. Since this limiting value of 0 is smaller than every value in our table, there is no global min.

Answer: Global max(es) at $t =$ $3+\sqrt{3}$ Global min(s) at $t =$ NONE

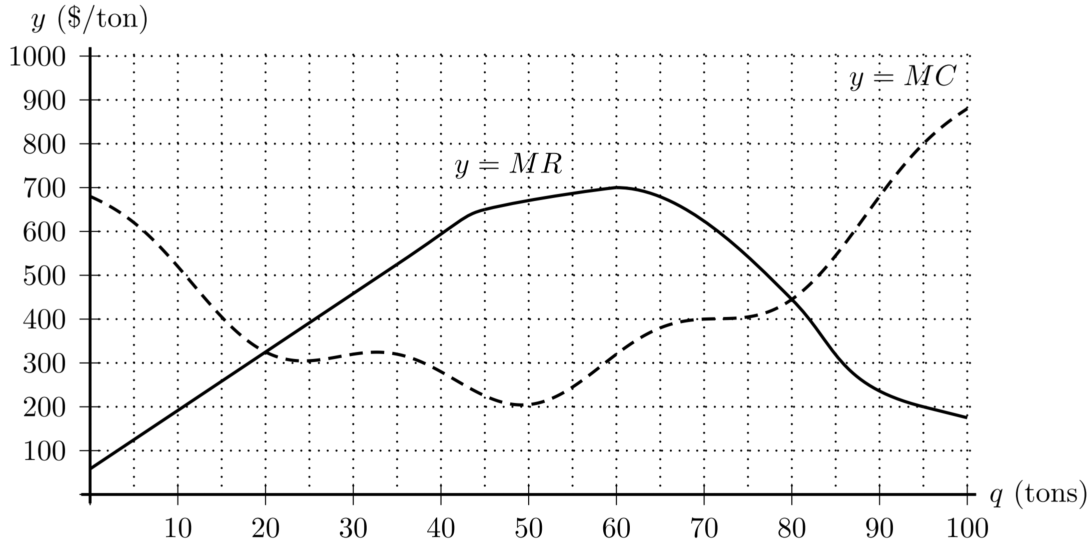
c. [3 points] Let $G(t)$ be the antiderivative of $g(t)$ with $G(0) = -5$. Find the t-coordinates of all critical points and inflection points of $G(t)$. For each answer black, write NONE if appropriate. You do not need to justify your answers.

Solution: Critical points of $G(t)$ are zeros of $g(t)$, of which there are none. Inflection points of $G(t)$ are local extrema of $g(t)$, which occur at $t = 3 \pm \sqrt{3}$ (which we know to be local extrema because they are in fact global extrema in the interiors of some intervals).

Answer: Critical point(s) at $t =$ NONE

Answer: Inflection point(s) at $t =$ $3-\sqrt{3}$, $3+\sqrt{3}$

9. [9 points] With winter past and summer approaching, David is opening a business selling ice. Graphed below are his marginal revenue MR (solid line) and marginal cost MC (dashed line), in dollars per ton of ice.



a. [4 points] Carefully estimate the answer to each of the following based on the graphs above. You do not need to show your work.

(i) For what value(s) of q in the interval $[0, 100]$ is revenue maximized?

Answer: $q =$ **100**

(ii) For what value(s) of q in the interval $[0, 100]$ is MR maximized?

Answer: $q =$ **60**

(iii) For what value(s) of q in the interval $[0, 100]$ is profit maximized?

Answer: $q =$ **80**

(iv) For what value(s) of q in the interval $[0, 100]$ is $MR - MC$ maximized?

Answer: $q =$ **50**

b. [2 points] David is planning to sell 5 tons of ice but is considering selling 35 tons instead.

(i) Would David's profit increase or decrease if he changed the amount of ice sold from 5 tons to 35 tons? (Circle one.)

INCREASE $\boxed{\text{DECREASE}}$

(ii) By how much would his profit increase or decrease? (Circle the one best estimate.)

\$1000 $\boxed{\$2000}$ \$4500 \$5250 \$6000

c. [3 points] Let $\pi(q)$ be David's profit, in dollars, if he sells q tons of ice. Suppose that David would make a profit of \$4000 if he sold 95 tons of ice. Find an equation for the tangent line to the graph of $y = \pi(q)$ at $q = 95$.

Solution: The slope of the tangent line is given by $\pi'(95)$, which we can read off the graph as the difference between MR and MC at $q = 95$, or about -600. Since the line passes through the point $(95, 4000)$, we therefore have the equation $y = 4000 - 600(q - 95)$.

Answer: $y = 4000 - 600(q - 95)$

10. [11 points] In each situation, circle <u>all</u> of the statements I-VI which <u>must</u> be true. If none of the statements must be true, circle VII. NONE OF THE ABOVE.

a. [3 points] Let $f(x) = qe^{rx} + s$, where q, r, and s are <u>negative</u> constants.

I. $f(0) > 0$

II. $\boxed{f'(0) > 0}$

III. $\boxed{\lim_{x\to\infty} f(x) = s}$

IV. $\lim_{x\to\infty} f(x) = 0$

V. $\lim_{x\to-\infty} f(x) = s$

VI. $\lim_{x\to-\infty} f(x) = 0$

VII. NONE OF THE ABOVE

b. [4 points] Let $g(x) = a\ln(bx)$, where a and b are <u>positive</u> constants.

I. $\boxed{\text{The domain of } g(x) \text{ is the interval } (0,\infty).}$

II. The graph of $g(x)$ has a horizontal asymptote.

III. $\boxed{\text{The graph of } g(x) \text{ has a vertical asymptote.}}$

IV. $\boxed{g^{-1}(0) = b^{-1}}$

V. $g'(x) = \dfrac{a}{bx}$

VI. $\boxed{\displaystyle\int g(x)\,dx = ax(\ln(bx) - 1) + C}$

VII. NONE OF THE ABOVE

c. [4 points] Let $z(t) = A\sin t + B$, where A and B are <u>positive</u> constants.

I. $\boxed{\text{The maximum value of } z(t) \text{ on its domain is } A + B.}$

II. $\boxed{z(t) \text{ has an inflection point at } t = 0.}$

III. $\boxed{\text{If } h(t) = z(z(t)), \text{ then } h'(0) = A^2\cos B.}$

IV. $\displaystyle\int_0^{2\pi} z(t)\,dt = 0$

V. $\boxed{\displaystyle\int_0^{\pi} z(t)\,dt = 2A + \pi B}$

VI. $\boxed{\displaystyle\int_1^{2} z(t)\,dt = \int_{1+2\pi}^{2+2\pi} z(t)\,dt}$

VII. NONE OF THE ABOVE