Math 115 — First Midterm

February 10, 2015

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: __________

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1. **Do not open this exam until you are told to do so.**
2. This exam has 11 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
8. Include units in your answer where that is appropriate.
9. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones.
10. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	8	
2	6	
3	10	
4	10	
5	14	
6	10	
7	10	
8	10	
9	12	
10	10	
Total	100	

1. [8 points] The table below gives several values of the continuous, invertible, differentiable functions $f(x)$ and $g(x)$.

x	1.8	1.9	2	2.1	2.2	2.3
$f(x)$	2.5	2.35	2.2	2	1.8	1.7
$g(x)$	1.6	1.75	1.8	1.9	2	2.2

a. [2 points] Compute $f(g^{-1}(2))$.

Solution: $f(g^{-1}(2)) = f(2.2) = 1.8$.

Answer: $f(g^{-1}(2)) =$ 1.8

b. [2 points] Estimate $f'(2)$.

Solution: We approximate using difference quotients. Using the average rate of change between $x = 1.9$ and $x = 2$ we have $f'(2) \approx \frac{2.2-2.35}{0.1} = -1.5$, and using the average rate of change between $x = 2$ and $x = 2.1$ we have $f'(2) \approx \frac{2-2.2}{0.1} = -2$. Averaging these two estimates, we find the estimate $f'(2) \approx -1.75$.

Answer: $f'(2) \approx$ -1.75

c. [2 points] Let $j(x) = g^{-1}(x)$. Estimate $j'(1.9)$.

Solution: A table of values for $j(x)$ is given by

x	1.6	1.75	1.8	1.9	2	2.2
$j(x)$	1.8	1.9	2	2.1	2.2	2.3

Estimating $j'(1.9)$ using the average rate of change between $x = 1.9$ and and $x = 2$ we find $j'(2) \approx \frac{2.2-2.1}{2-1.9} = 1$. (We obtain the same estimate using the interval from $x = 1.8$ to $x = 1.9$.)

Answer: $j'(1.9) \approx$ 1

d. [2 points] Suppose $p(x)$ is a function whose derivative is given by $p'(x) = \ln(x^3 + 11)$. Compute $p'(f(2))$.

Solution: $p'(f(2)) = p'(2.2) = \ln((2.2)^3 + 11) = \ln(21.648) \approx 3.0749$

Answer: $p'(f(2)) =$ $\ln(21.648)$

2. [6 points] Suppose a and b are constants with $a > 3$ and $b > 0$, and let $h(t) = a^{-bt}$.

a. [3 points] Find constants P_0 and k so that $h(t) = P_0e^{kt}$. (Your answers may involve the constants a and/or b.)

Solution: If $P_0e^{kt} = a^{-bt}$ for all t, then $P_0 = 1$ and $a^{-b} = e^k$, so $k = \ln(a^{-b}) = -b\ln(a)$. Alternatively, we can directly rewrite the original formula as $h(t) = a^{-bt} = (e^{\ln a})^{-bt} = e^{-b\ln(a)t}$.

Answer: $P_0 =$ 1 and $k =$ $-b\ln(a)$

b. [3 points] Circle <u>all</u> the statements below that <u>must</u> be true about the function $h(t)$. If none of the statements must be true, circle NONE OF THESE.

Solution: Since $a > 1$ and $b > 0$, $h(t)$ is a decreasing exponential function.

i. [✓ The domain of $h(t)$ is the interval $(-\infty, \infty)$.]

ii. The range of $h(t)$ is the interval $(-\infty, \infty)$.

iii. $h(t)$ is an increasing function on its domain.

iv. $h(t)$ [✓ is concave up on its domain.]

v. $t = 0$ is a vertical asymptote of the graph of $h(t)$.

[✓ vi. $\lim_{t\to\infty} h(t) = 0$.]

vii. $\lim_{t\to-\infty} h(t) = 0$.

NONE OF THESE

3. [10 points] Elphaba the squirrel has been involved in some questionable activity of late and hence is being very cautious. She has made eye contact with a human standing near her multiple times and is getting anxious that the human is observing her. Let $f(x)$ be Elphaba's anxiety (in "anxious units") after making eye contact with the human for a total of x seconds. Elphaba will panic and run when her anxiety reaches 100 anxious units.

From across the room, the human, Erin, is in fact observing Elphaba while pretending to read a newspaper. The total amount of time Elphaba has spent making eye contact with Erin is a function of the number of times that Erin looks up from the newspaper. Let $g(n)$ be the total amount of time, in seconds, that Erin and Elphaba have spent making eye contact if Erin has looked up from her newspaper n times.

a. [2 points] Using a complete sentence, give a practical interpretation of the expression $f^{-1}(3) = 10$. Be sure to include units.

Solution: After Elphaba has made eye contact with Erin for a total of 10 seconds, her anxiety is at 3 anxious units.

Alternative: Elphaba's anxiety is at 3 anxious units when she has made eye contact with Erin for a total of 10 seconds.

b. [3 points] Below is the first part of a sentence that will give a practical interpretation of the equation $$f'(25) = 2\,.$$

Complete the sentence so that the practical interpretation can be understood by someone who knows no calculus. Be sure to include units in your answer.

If Elphaba has already made eye contact with Erin for a total of 25 seconds and she makes eye contact for an additional 0.3 seconds, then

Solution: If Elphaba has already made eye contact with Erin for a total of 25 seconds and she makes eye contact for an additional 0.3 seconds, then Elphaba's anxiety will increase by approximately 0.6 anxious units.

(Note: $0.6 = 2 \times 0.3$)

c. [2 points] Given that $(f^{-1})'(99) = 7$ and $f(62) = 99$, approximate the total length of time Elphaba has to spend making eye contact with Erin before she will panic and run.

Solution: The expression $(f^{-1})'(99) = 7$ means that once Elphaba's anxiety is at 99 anxiety units, it takes approximately 7 seconds for her anxiety to reach 100 anxious units. The expression $f(62) = 99$ means that it takes 62 seconds for Elphaba's anxiety to reach 99 anxious units. Therefore putting these together, it will take approximately $62 + 7 = 69$ seconds for Elphaba's anxiety to reach 100 anxious units, which is when she will panic and run.

d. [3 points] Which of the following sentences gives a correct interpretation of the quantity $g^{-1}(f^{-1}(50))$? *Circle the* ONE *best answer.*

i. When Erin has looked up from her newspaper 50 times, Elphaba's anxiety is at $g^{-1}(f^{-1}(50))$ anxious units.

ii. When Erin has looked up from her newspaper 50 times, Erin and Elphaba have spent $g^{-1}(f^{-1}(50))$ seconds making eye contact.

iii. [✓ If Erin has looked up from her newspaper $g^{-1}(f^{-1}(50))$ times then Elphaba's anxiety is 50 anxious units.]

iv. If Erin and Elphaba have made eye contact for a total of 50 seconds then Erin has looked up from her newspaper $g^{-1}(f^{-1}(50))$ times.

v. When Erin and Elphaba have made eye contact for a total of 50 seconds then Elphaba's anxiety is at 50 anxious units.

4. [10 points] For each of the following, give a *formula* for a single function satisfying all of the listed properties. If there is no function satisfying all the properties, circle NO SUCH FUNCTION EXISTS.
Note: If "NO SUCH FUNCTION EXISTS" is circled, then any formula you have written will not be graded.

a. [3 points] A *polynomial* $p(t)$ with the following three properties:

- The degree of $p(t)$ is three.
- $p(t) \to -\infty$ as $t \to \infty$.
- $p(0) = -4$.

Solution: Note that the second property implies that the leading coefficient of the polynomial is negative, and the third property implies that, when written in standard form, the constant term of $p(t)$ is -4. So one example is $p(t) = -t^3 - 4$.

Answer: $p(t) =$ $-t^3 - 4$ OR Circle: NO SUCH FUNCTION EXISTS

b. [3 points] An *exponential function* $q(v)$ with the following three properties:

- $q(1) = 3$.
- $\lim_{v \to 0} q(v) = 12$.
- $\lim_{v \to \infty} q(v) = 0$.

Solution: Since exponential functions are continuous, the second property implies that $q(0) = 12$. So $q(v)$ is exponential with initial value 12 and decay factor equal to $\frac{q(1)}{q(0)} = \frac{3}{12} = \frac{1}{4}$. Therefore q must be the function given by $q(v) = 12\left(\frac{1}{4}\right)^v$.

Answer: $q(v) =$ $12\left(\frac{1}{4}\right)^v$ OR Circle: NO SUCH FUNCTION EXISTS

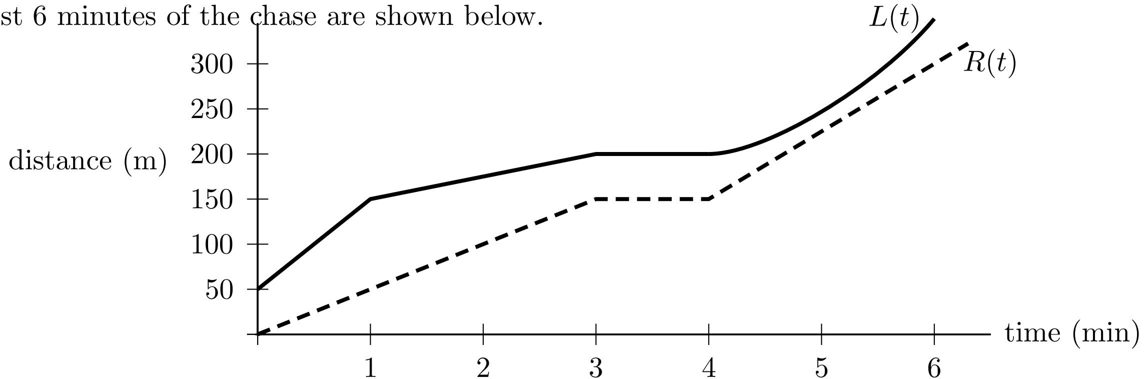
c. [4 points] A *rational* function $r(x)$ with the following three properties:

- The line $x = 2$ is a vertical asymptote of the graph of $y = r(x)$.
- The line $y = -3$ is a horizontal asymptote of the graph of $y = r(x)$.
- $r(5) = 0$.

Solution: These properties imply that $r(x)$ can be written as a quotient of polynomials $\frac{p(x)}{q(x)}$ such that $(x-2)$ is a factor of $q(x)$, the ratio of the leading term of $p(x)$ to that of $q(x)$ is -3, and $(x-5)$ is a factor of $p(x)$. There are many possibilities, but one example is $r(x) = \frac{-3(x-5)}{x-2}$.

Answer: $r(x) =$ $\dfrac{-3(x-5)}{x-2}$ OR Circle: NO SUCH FUNCTION EXISTS

5. [14 points] Elphaba the squirrel is panicking because she has noticed that a human, Erin, is watching her. Elphaba starts to run and Erin is soon in full-blown pursuit as they both run straight down the street. Let $R(t)$ be Erin's distance from their starting point (in meters) t minutes after the chase begins and $L(t)$ be Elphaba's distance from the starting point (in meters) t minutes after the chase begins. The graphs of $R(t)$ (dashed) and $L(t)$ (solid) for the first 6 minutes of the chase are shown below.



a. [1 point] Which of the following expressions gives the distance, in meters, between Elphaba and Erin t minutes after the chase begins? *Circle the* ONE *best option.*

i. $L'(t) - R'(t)$ ii. $R'(t) - L'(t)$ iii. [✓ $L(t) - R(t)$] iv. $R(t) - L(t)$ v. $R^{-1}(L(t))$ vi. $L^{-1}(R(t))$

b. [2 points] What is Erin's velocity when $t = 0.5$? *Be sure to include units.*

Solution: We are looking for the slope of the dotted line that represents Erin's distance at time $t = 0.5$. This is $\frac{150-0}{3-0} = 50$. So Erin is travelling at 50 m/min.

Answer: 50 **m/min**

c. [3 points] During which of the following time periods is Erin gaining on Elphaba? *Circle* ALL *correct answers.*

i. $0 \le t \le 0.75$ ii. [✓ $1.25 \le t \le 2.75$] iii. $3.25 \le t \le 3.75$ iv. [✓ $4.25 \le t \le 4.75$] v. $5.25 \le t \le 6$

Solution: Erin is gaining on Elphaba when Erin is travelling faster than Elphaba. So we are looking for time periods when the slope of Erin's graph is steeper than the slope of Elphaba's graph.

d. [3 points] During which of the following time periods is there at least one time when Erin and Elphaba are travelling at the same speed? *Circle* ALL *correct answers.*

Solution: Erin and Elphaba are travelling at the same speed when the slopes of their distance graphs are equal.

i. $0.25 \le t \le 0.75$ ii. $1.75 \le t \le 2.25$ iii. $2.25 \le t \le 2.75$ iv. [✓ $3.25 \le t \le 3.75$] v. [✓ $4.75 \le t \le 5.25$]

e. [2 points] Circle all of the following events that could be occurring between the 3rd and the 4th minutes.

Solution: Neither Erin nor Elphaba is moving during this time.

i. Elphaba is getting further from Erin.

iii. [✓ Elphaba has stopped.]

ii. [✓ Erin is tying her shoe.]

iv. Erin is gaining on Elphaba.

f. [3 points] What is Elphaba's average velocity over the first 3 minutes of the chase? *Be sure to include units.*

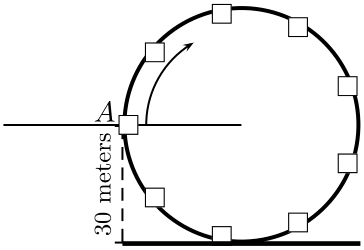
Solution: Elphaba's average velocity over the first 3 minutes is given by

$$\frac{L(3) - L(0)}{3-0} = \frac{200-50}{3} = 50 \text{ m/min}.$$

Answer: 50 **m/min**

6. [10 points] Erin is in pursuit of squirrel and suspected criminal, Elphaba. Suddenly there is a cliff ahead. Elphaba, who it turns out is a flying squirrel, jumps straight off and glides safely down to the ground. Searching for an alternative, Erin finds a ferris wheel that will take her to the ground beneath the cliff.

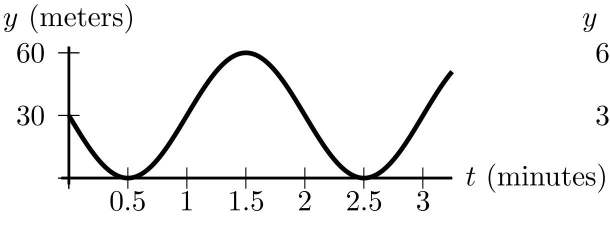
The ferris wheel has radius 30 meters and is rotating (clockwise in the diagram shown) at a constant rate of one half radian per minute. Let $H(t)$ be Erin's height above the ground beneath the cliff (in meters) t minutes after she gets on the ferris wheel. A diagram of the situation is shown to the right. Note that Erin gets on the ferris wheel at position A, and $H(0) = 30$.



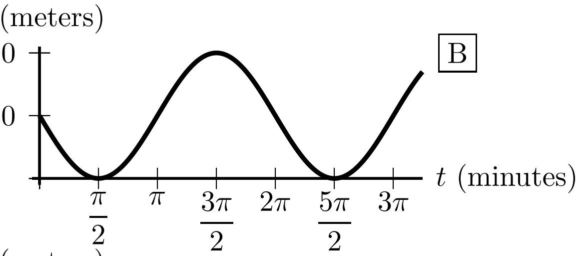
a. [2 points] Which of the following graphs is a graph of $y = H(t)$?
Write the letter (A-D) of the ONE *best choice.* **Answer:** C

Solution: Erin is 30 meters above the ground at time $t = 0$, and her height above the ground increases at first. Since the ferris wheel is rotating at a constant rate of $1/2$ radian per minute, it will take 4π minutes for it to rotate 2π radians. So the period of $H(t)$ is 4π. The graph shown as C below is the best choice.

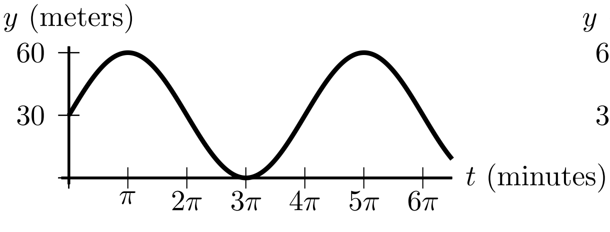
A



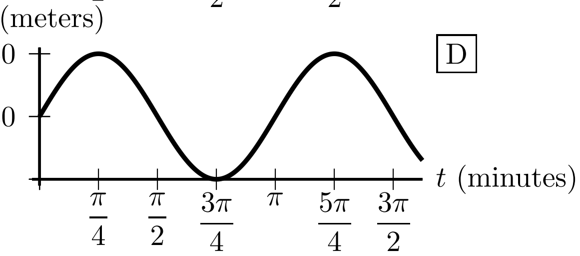
B



C



D



b. [4 points] Write a formula for the sinusoidal function $H(t)$.

Solution: $H(t)$ is a sinusoidal function with amplitude 30, midline $y = 30$, and period 4π. The value is on the midline and increasing when $t = 0$. There are many possible formulas. One is $H(t) = 30\sin(\frac{t}{2}) + 30$.

Answer: $H(t) =$ $30\sin\left(\frac{t}{2}\right) + 30$

c. [4 points] Erin figures that if she jumps off when she is no more than b meters above the ground, where b is a constant between 0 and 30, then she will be fine. Erin would like to jump off before she has to go around the ferris wheel again. What is the latest time she can jump off without going around a full revolution? Remember to show your work clearly. Your answer may involve the constant b.

Solution: We are looking for a particular solution to the equation $30\sin(t/2) + 30 = b$. Using inverse trig to solve this, we find that one solution is $t = 2\sin^{-1}(\frac{b-30}{30})$. Now we need to think about what point on the graph this actually gives us. In this case, since $-1 < \frac{b-30}{30} < 0$ this solution is the first solution to the left of the vertical axis in the graph above. We need to add the period 4π to get the time we are looking for. So to avoid going around a full revolution, the latest time Erin can jump off is $2\sin^{-1}\left(\frac{b-30}{30}\right) + 4\pi$ minutes after she got on the ferris wheel.

Answer: $t = 2\sin^{-1}\left(\frac{b-30}{30}\right) + 4\pi$

7. [10 points] Sebastian has chartered a helicopter which is rising straight up in the air, but he is scared of heights. Let $A(w)$ be Sebastian's fear (in "scared units") when he is w km above the ground. For $0 < w \leq 2$, a formula for $A(w)$ is given by

$$A(w) = \frac{w^2+2}{w^w+1}.$$

a. [5 points] Use the limit definition of the derivative to write an explicit expression for the instantaneous rate of change of Sebastian's fear, in scared units per km, when he is 1.5 km above the ground. *Your answer should not involve the letter A. Do not attempt to evaluate or simplify the limit.*

Solution: The instantaneous rate of change of Sebastian's fear, in scared units per km, is given by the derivative

$$A'(1.5) = \lim_{h \to 0} \frac{A(1.5+h) - A(1.5)}{h}.$$

Answer: $A'(1.5) = \boxed{\lim_{h \to 0} \frac{\frac{(1.5+h)^2+2}{(1.5+h)^{(1.5+h)}+1} - \frac{1.5^2+2}{1.5^{1.5}+1}}{h}}$

b. [5 points] When he has reached a height of 2 km above the ground Sebastian gets control of his fear and his fear starts decreasing at a constant rate of 0.8 scared units per km. Write a formula for a piecewise-defined continuous function $A(w)$ giving Sebastian's fear, in scared units, for $0 < w < 3$.

Solution: We were given that $A(w) = \frac{w^2+2}{w^w+1}$ for $0 < w \leq 2$. So it remains to find a formula for $A(w)$ that is valid for $2 < w < 3$. Since Sebastian's fear is decreasing at a constant rate for $w > 2$, $A(w)$ is linear for $2 < w < 3$, and the slope of this linear piece is -0.8. In order for $A(w)$ to be continuous, this linear piece must pass through the point $(2, A(2))$ which is $(2, 6/5)$. Using point slope form, this gives the formula $1.2 - 0.8(w-2) = 2.8 - 0.8w$ for the linear piece.

Answer: $A(w) = \begin{cases} \dfrac{w^2+2}{w^w+1} & \text{if } 0 < w \leq 2 \\ 2.8 - 0.8w & \text{if } 2 < w < 3. \end{cases}$

8. [10 points] *Throughout this page, give all answers in exact form. Do not use decimal approximations. For example,* $x = \frac{1}{3}$ *is an exact solution to* $3x = 1$*, but* $x = 0.3333333333$ *is not.*

Sebastian has rented a helicopter to catch up to his friend Erin who is currently chasing a suspected criminal named Elphaba. When Sebastian first sees the pair they are 180 meters apart. After 3 minutes, Erin has moved 60 meters closer to Elphaba. (In other words, the distance between them has decreased by 60 meters.) Let $D(t)$ be the distance between Elphaba and Erin, in meters, t minutes after Sebastian begins watching them.

a. [2 points] Sebastian initially assumes that $D(t)$ is a linear function. Find a formula for $D(t)$ under this assumption, valid for as long as it takes for Erin to catch Elphaba.

Solution: If $D(t)$ is linear, then $D(t) = b + mt$ where b is the initial value and m is the constant average rate of change. We immediately know $b = 180$ since this is the initial value. We then know the function decreases by 60 m in 3 minutes so that's an average rate of decrease of 20m/min. Therefore $D(t) = 180 - 20t$.

Answer: $D(t) =$ $180 - 20t$

b. [1 point] After Sebastian has been watching for 6 minutes, the distance between Erin and Elphaba is 80 meters. Briefly explain why this contradicts Sebastian's initial assumption.

Solution: One way to see is that the point $(6, 80)$ doesn't satisfy the formula we found in a. Another explanation would be that the average rate of change between $(0, 180)$ and $(3, 120)$ is different from the average rate of change between $(3, 120)$ and $(6, 80)$.

c. [4 points] Sebastian then determines that $D(t)$ must in fact be an exponential function. Write a new formula for $D(t)$ given this new information (including the data from part (b)). *Remember to show your work carefully and use exact form.*

Solution: We now know $D(t) = bc^t$ where b is again the initial value, so $D(t) = 180c^t$. We can solve for c by using the fact that $D(3) = 120$. Then $120 = 180c^3$ so $c = (\frac{120}{180})^{1/3} = \left(\frac{2}{3}\right)^{1/3}$.

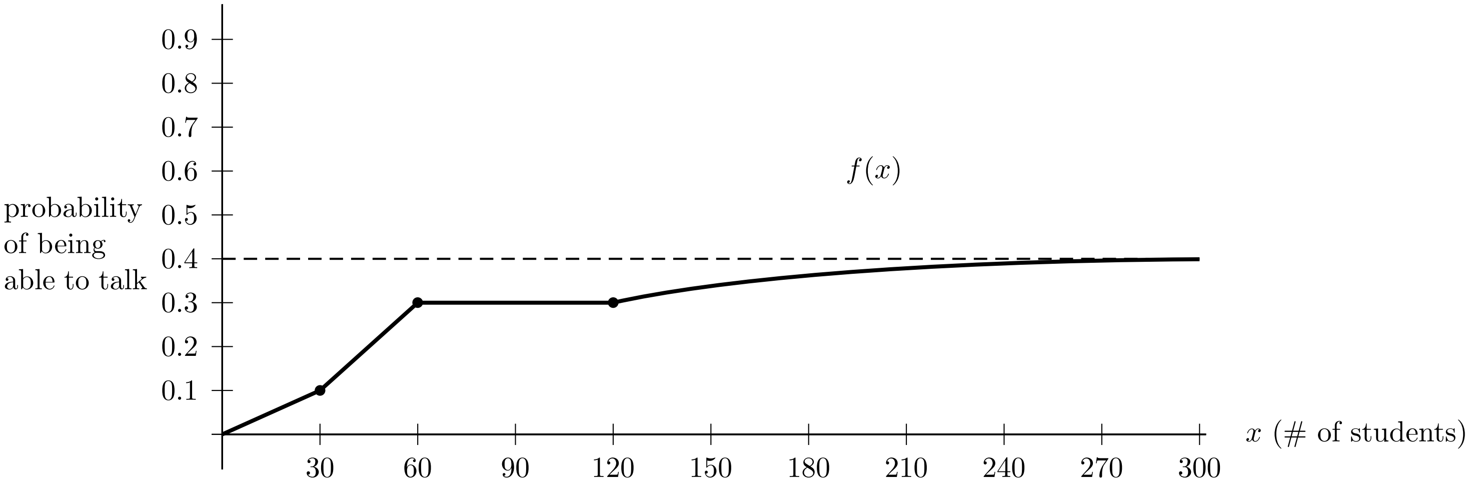
Answer: $D(t) =$ $180((\frac{120}{180})^{\frac{1}{3}})^t = 180\left(\frac{2}{3}\right)^{t/3}$

d. [3 points] Erin can catch Elphaba when she is within one meter of her (since Erin can jump and tackle Elphaba at this distance). Use algebra and your formula from part (c) to find how long it takes for the distance between Erin and Elphaba to decrease to 1 meter.

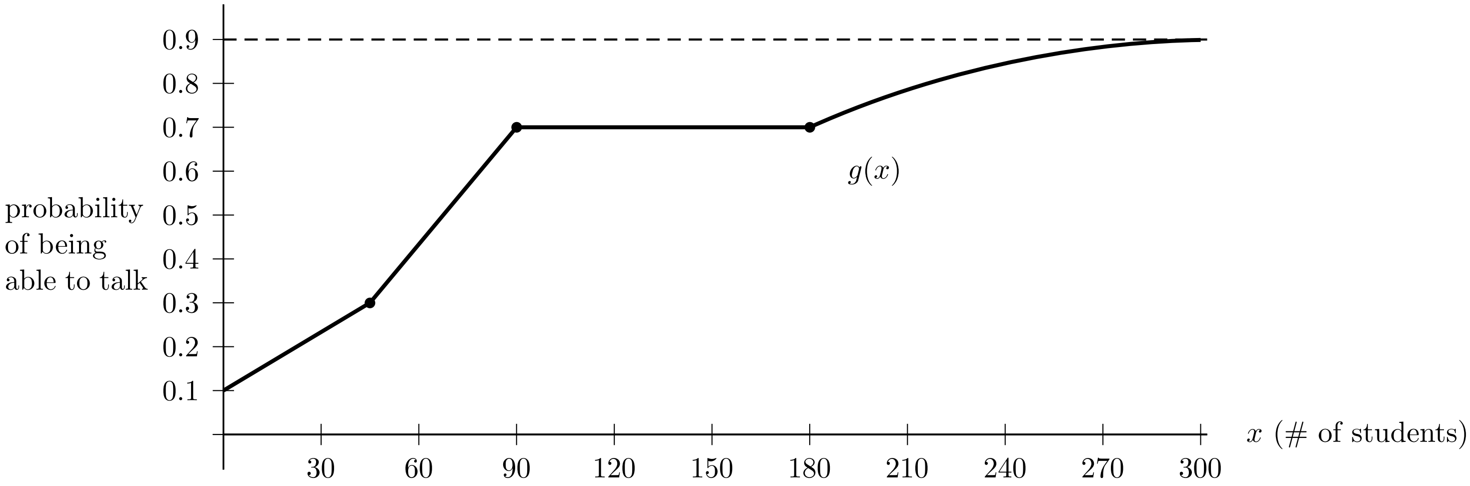
Solution: We need to solve for t in the equation $D(t) = 1$. Using our formula from part (c), we have $180((\frac{2}{3})^{\frac{1}{3}})^t = 1$. Solving, we find $((\frac{2}{3})^{\frac{1}{3}})^t = \frac{1}{180}$ so $t\ln((\frac{2}{3})^{\frac{1}{3}}) = \ln(\frac{1}{180})$. Finally $t = \frac{\ln(\frac{1}{180})}{\ln((\frac{2}{3})^{\frac{1}{3}})} = \frac{-3\ln(180)}{\ln(2)-\ln(3)} = \frac{3\ln(180)}{\ln(1.5)}$. So it takes $\frac{3\ln(180)}{\ln(1.5)}$ (about 38.4) minutes for Erin to get within 1 meter of Elphaba.

Answer: $\dfrac{\ln\left(\frac{1}{180}\right)}{\ln\left(\left(\frac{2}{3}\right)^{\frac{1}{3}}\right)} = \dfrac{3\ln(180)}{\ln(1.5)}$

9. [12 points] Lauren has just been approached by a talking kangaroo named Skipper. Lauren is alarmed by Skipper's ability to talk so she asks him if all kangaroos can talk. Skipper tells her that a kangaroo's probability of being able to talk depends on how much attention they receive from University of Sydney students. He also explains that this relationship has changed recently. Let $f(x)$ be the probability that a kangaroo born prior to 5 weeks ago is able to talk if x students have paid attention to that kangaroo in their life. Below is a graph of the function $f(x)$.



a. [4 points] Let $g(x)$ be the probability that a kangaroo born within the last 5 weeks is able to talk if x students have paid attention to that kangaroo in their life. A graph of $g(x)$ is given below.



It turns out that for $x \geq 0$, $g(x)$ can be expressed as a transformation of $f(x)$. Write a formula for $g(x)$ as a transformation of $f(x)$.

Solution: To obtain the graph of $g(x)$ from that of $f(x)$, we first vertically stretch by a factor of 2, then shift the resulting graph up by 0.1 units, and finally stretch it horizontally by a factor of 3/2.

Answer: $g(x) = 2f\left(\frac{2}{3}x\right) + 0.1$

Problem continues on the next page.

This is a continuation of the problem from the previous page.

b. [3 points] Let $h(y)$ be the number of students that pay attention to a kangaroo over the duration of the kangaroo's life if the kangaroo has y docile units. (Note that we measure dorileness in terms of docile units.) Give a practical interpretation of $g(h(5))$.

Solution: $g(h(5))$ is the probability that a kangaroo born in the last 5 weeks with 5 docile units will be able to talk.

c. [3 points] The number of students who pay attention to a kangaroo with y docile units is proportional to y^2. Find a formula for $h(y)$ if a total of 160 students pay attention to a kangaroo with 4 docile units during its life.

Solution: The first sentence tells us that $h(y) = ky^2$ for some constant k. Using the fact that $h(4) = 160$, we find $160 = k(4^2)$, so $160 = 16k$ and $k = 10$.

Answer: $h(y) =$ $10y^2$

d. [2 points] Calculate $f(h(3))$.

Solution: Using our formula from part c, we have $h(3) = 10(3^2) = 90$ so $f(h(3)) = f(90) = 0.3$.

Answer: $f(h(3)) =$ 0.3

10. [10 points] On the axes provided below, sketch the graph of a single function $y = h(x)$ satisfying all of the following:

- $h(x)$ is defined for all x in the interval $-6 < x < 6$.
- $h'(x) < 0$ for all $x < -3$.
- $\lim\limits_{x \to -2^+} h(x) = -1$.
- $h'(0) = 0$.
- The average rate of change of $h(x)$ between $x = -1$ and $x = 2$ is 1.
- $h(x)$ is not continuous at $x = 3$.
- $h(x) > 0$ for all $x > 3$.
- $h'(x) > 0$ for all $x > 4$.

Make sure that your sketch is large and unambiguous.