

PRACTICE FINAL EXAM A

PART I (6 pts each) *Answer any 17 of the following 21 questions.* You need not justify your answer. You may answer more than 17 to obtain extra credit.

1. $\lim_{n \rightarrow \infty} \frac{(2n+1)^3(n+2017)^5}{n(4n+3)(n-2017)^7}$

2. $\lim_{x \rightarrow 0} \frac{\cos x - 1}{e^x - x^2 - x - 1}$

3. $\lim_{x \rightarrow 0} \frac{\ln(ax+1)}{\ln(bx+1)}$ where a and b are positive constants.

4. Let $h(x) = \int_1^x \ln(1 + 2017 \ln t) dt$

Compute $h'(e)$.

5. $\frac{d^{2017}}{dx^{2017}} \sin(7x) =$

6. Solve the initial value problem:

$$\frac{dy}{dx} = \frac{\ln x}{x} \quad \text{given that } y = 2017 \text{ when } x = 1.$$

7. Find an anti-derivative of $\frac{(1 + \sqrt{x})^{\frac{4}{5}}}{\sqrt{x}}$

8. Find an anti-derivative of:

$$\frac{1 + 3e^{3x} - e^{-x}}{e^{3x} + e^{-x} + x}$$

9. $\lim_{x \rightarrow \infty} x \tan\left(\frac{1789}{x}\right) =$

10. Suppose that $\int_7^{13} f(x) dx = 3$ and $\int_7^{13} g(x) dx = 1$.

Find $\int_7^{13} (4f(x) - 3g(x) + 2) dx$

11. Find the *average value* of the function $y = \sec^2 x$ over the interval $[0, \pi/4]$. (Give the precise result without rounding.)

12. Find the value of c such that the conclusion of the Mean Value Theorem is verified for the function $g(x) = \frac{1}{(x-1)^2}$ on the interval $[2, 5]$. Express your answer to the nearest hundredth.

13. Find $\lim_{n \rightarrow \infty} \left(\sqrt{n^2 + 92n + 2017} - n \right)$

14. Let a and b be non-zero constants. Let $f(x) = \frac{x+a}{x^2+b}$.

Find the *slope of the tangent line* to $y = f(x)$ at $x = 0$. (Your answer may include the constants a and b .)

15. Let a and b be non-zero constants. Then $\int \frac{\sec x \tan x}{a + b \sec x} dx =$

16. Suppose that $\int_1^x g(t) dt = x^3 - 1$. Find $g(5)$.

17. Let $y = x^{x^2+x+1}$. Find dy/dx when $x = 1$.

18. Compute $\int_{-1}^3 |x| dx$.

19. Compute $\lim_{n \rightarrow \infty} \sum_{j=1}^n \frac{1}{n} \sin\left(\frac{j}{n} \pi\right)$ (Hint: Convert this limit into a Riemann integral.)

20. Given that $G(x) = \int_0^{3x^2} \sqrt{1+t^3} dt$, find $G'(1)$.

21. Charlotte, the spider, lives on the x-axis. Suppose that at time $t = 1$ minute, she is at $x = 5$ cm, and that her velocity (in cm/minute) at time t is given by:

$$v(t) = 4t^3 - 6t^2 + 1. \quad \text{Where is Charlotte at time } t = 2 \text{ minutes?}$$

PART II (12 pts each)

Answer any 13 of the following 14 problems. You may answer more than 11 for extra credit.

1. Find the equation of the *tangent line* to the curve defined implicitly by

$$x^4 + y^3 - x^2y = 13 + \ln y$$

at the point $P = (2, 1)$.

2. Gilberte, who is 5 feet tall, walks away from an 18 foot lamppost. She observes that when she is 8 feet from the base of the lamppost, her shadow is increasing at a rate of 6 ft/min. Find Gilberte's speed when she is 8 feet from the base of the lamppost.

3. Using an *appropriate tangent line approximation*, estimate the value of $\sqrt[5]{1.0004}$. Have you obtained an *overestimate* or an *underestimate*? Explain. Sketch!

4. We wish to approximate a root of $g(x) = x^4 + x - 1$. Note that $g(0) < 0$ and $g(1) > 0$.

(a) How do we know that there must exist a solution to $g(x) = 0$ in the interval $(0, 1)$?

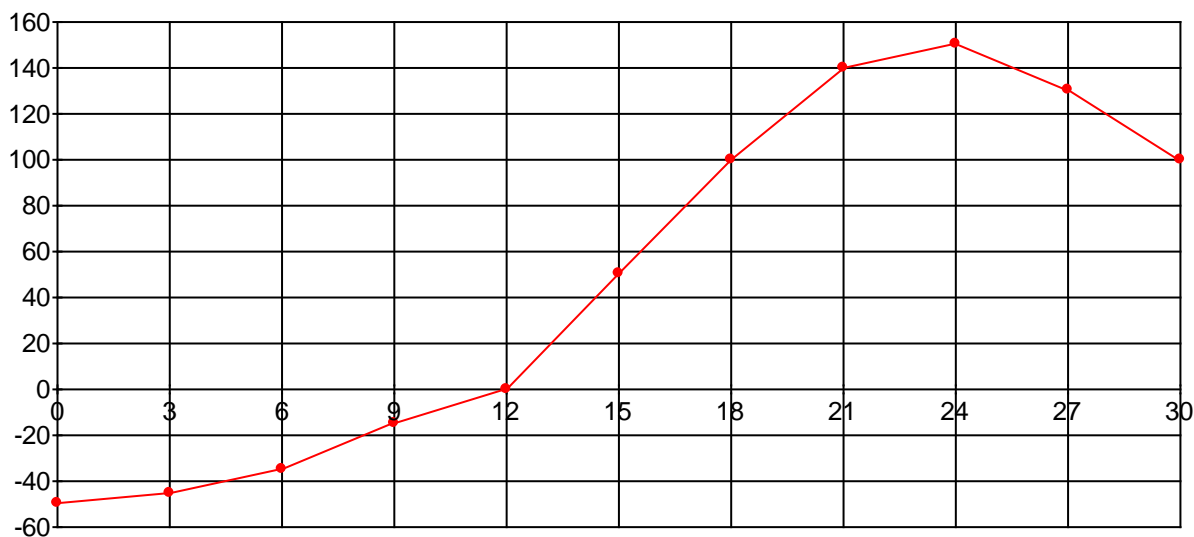
(b) Let our initial guess for the root be $x_0 = 0.5$. Using Newton's method, compute x_1 and x_2 (to at least 3 significant digits).

5. Graph the function $f(x) = (x-1)^2 e^x$. Identify any and all local and global extrema and points of inflection.

6. Madam Verdurin is building an open planter in the shape of a rectangular box with a square base. The base is made of metal that costs \$7 per square foot. The sides are made of wood that costs \$3 per square foot. The planter must hold at least 8 cubic feet of dirt. Find the dimensions of the *least expensive* planter that Madame Verdurin can build.

7. The graph below shows the *RATE OF CHANGE* of the quantity of water in the Water Tower of OZ, in liters per day, during the month of April, 2009. The tower contained 12,000 liters of water on April 1. *Estimate* the quantity of water in the tower on April 30. Show your work.

Rate of Change of Quantity of Water



8. Using the FTC, find the area bounded by the two parabolas:

$$y = x^2 - 5x \text{ and } y = 20 + x - x^2. \text{ Sketch.}$$

9. Use a *left-endpoint* Riemann sum with $n = 4$ rectangles to approximate the area under the curve $f(x) = \frac{1}{x^3 + 1}$ between $x = 0$ and $x = 2$. Draw a picture to illustrate what you are computing. Is this an *underestimate* or an *overestimate* of the area? *Explain!*

10. The function $y = F(x)$ is defined below:

$$F(x) = \begin{cases} \frac{3x^4 - 2x^3 - 21x^2}{x-3} & \text{if } x \neq 3 \\ k & \text{if } x = 3 \end{cases}$$

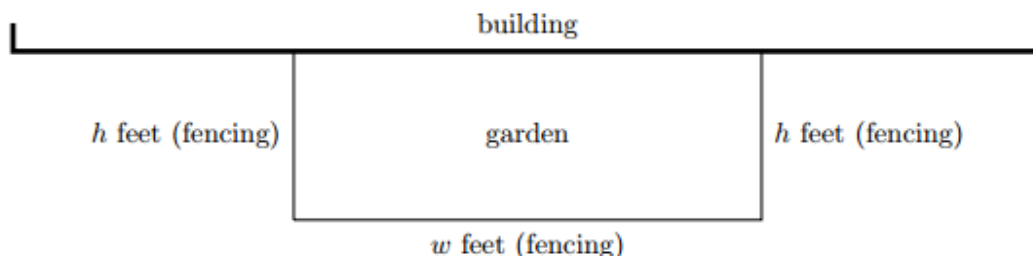
For which value(s) (if any) of k is the function everywhere continuous? Explain!

11. Graph the cubic polynomial $g(x) = x^3 + x^2 - 8x + 5$. Identify any and all local and global extrema and points of inflection.

12. (*University of Michigan*)

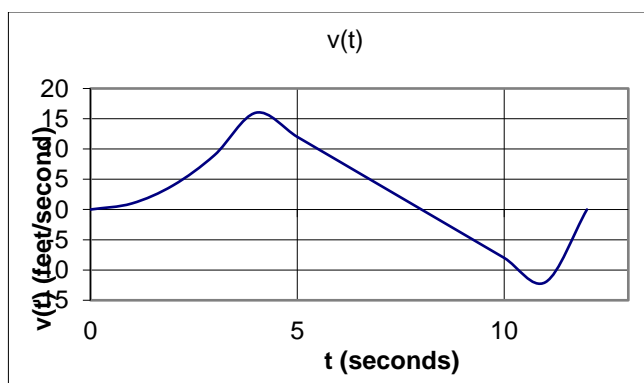
[12 points] Researchers are constructing a rectangular garden adjacent to their building. The garden will be bounded by the building on one side and by a fence on the other three sides. (See diagram below.) The fencing will cost them \$5 per linear foot. In addition, they will also need topsoil to cover the entire area of the garden. The topsoil will cost \$4 per square foot of the garden's area.

Assume the building is wider than any garden the researchers could afford to build.



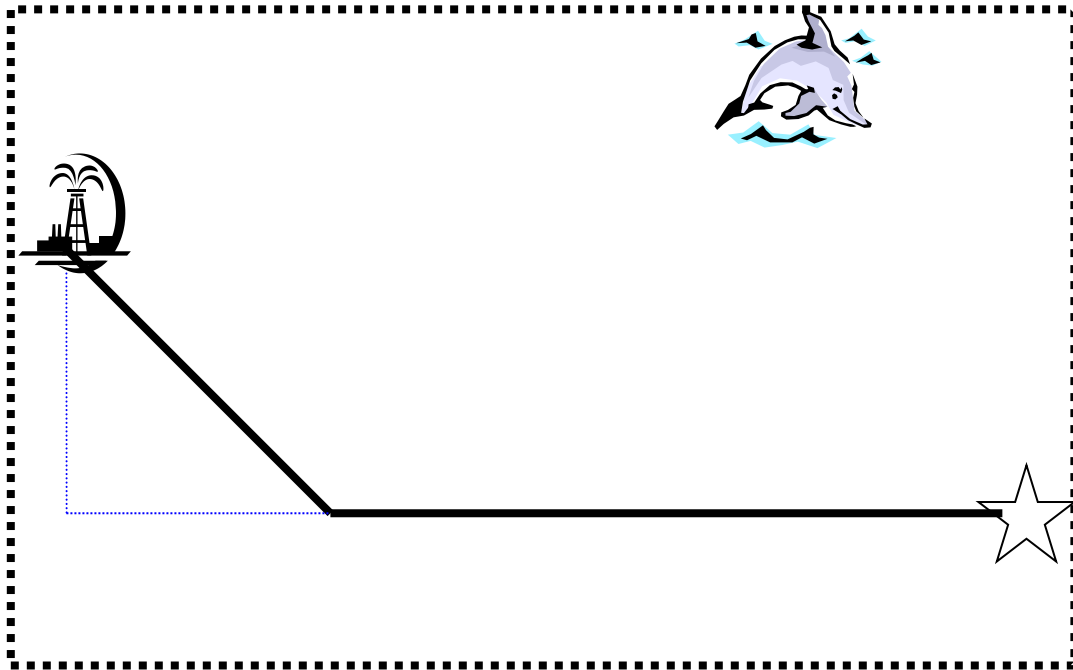
- a. [5 points] Suppose the garden is w feet wide and extends h feet from the building, as shown in the diagram above. Assume it costs the researchers a total of \$250 for the fencing and topsoil to construct this garden. Find a formula for w in terms of h .
- b. [3 points] Let $A(h)$ be the total area (in square feet) of the garden if it costs \$250 and extends h feet from the building, as shown above. Find a formula for the function $A(h)$. The variable w should not appear in your answer.
(Note that $A(h)$ is the function one would use to find the value of h maximizing the area. You should not do the optimization in this case.)
- c. [4 points] In the context of this problem, what is the domain of $A(h)$?

13. Albertine launches a model rocket from the ground at time $t = 0$. The rocket starts by traveling straight up in the air. The graph below illustrates the upward velocity of the rocket as a function of time.



- (a) Sketch a graph of the *acceleration* of the rocket as a function of time.
- (b) Sketch a graph of the *height* of the rocket as a function of time.
- (c) Give an estimate of the *maximum height* the rocket achieved.

14. Oil from an offshore rig located 3 miles from the shore is to be pumped to a location on the edge of the shore that is 9 miles east of the rig. The cost per foot of constructing a pipe in the ocean from the rig to the shore is *twice the cost per foot* of construction on land. Determine how the pipe should be laid to *minimize the total cost*.



With an absurd oversimplification, the "invention" of the calculus is sometimes ascribed to two men, Newton and Leibniz. In reality, the calculus is the product of a long evolution that was neither initiated nor terminated by Newton and Leibniz, but in which both played a decisive part.

- Richard Courant and Herbert Robbins

The quarrel [between Newton and Leibniz] is simply the expression of evil weaknesses and fostered by vile people. Just what would Newton have lost if he had acknowledged Leibniz's originality? Absolutely nothing! He would have gained a lot. And yet how hard it is to acknowledge something of this sort: someone who tries it feels as though he were confessing his own incapacity. ... It's a question of envy of course. And anyone who experiences it ought to keep on telling himself: "It's a mistake! It's a mistake! -- "

- Ludwig Wittgenstein (1947)