1. Albertine is studying Newton's method. She is trying to find the roots of a cubic polynomial

$$p(x) = x^3 - x + 1 = 0$$

(a) *[3 pts]* Albertine has determined that there must be a root between x = -2 and x = -1. How can she be so certain?

Solution: First observe that p(x) is continuous, since it is a polynomial.

Then note that p(0) = 1 > 0 and p(-2) = -5 < 0.

Hence, by the Intermediate Value Theorem, p(x) must have a root in the interval (0, -2).

(b) *[7 pts]* Albertine's initial guess is $x_0 = -1$. Using Newton's method, find x_1 and x_2. Express each answer to 3 significant digits.

Solution: Letting $x_0 = -1$, *we find that* $y_0 = f(x_0) = f(-1) = 1$.

Also $p'(x) = 3x^2 - 1 \Rightarrow p'(x_0) = p'(-1) = 2$.

So the equation of the tangent line to y = p(x) at $x = x_0$ *is*

$$y - 1 = 2(x - (-1))$$

Letting y = 0, we find: $x_1 = -\frac{1}{2} - 1 = -\frac{3}{2}$.

Repeating this process to find x_2 *;*

$$y_1 = p\left(-\frac{3}{2}\right) = -0.875$$

$$m_1 = p'\left(-\frac{3}{2}\right) = 5.75$$

So the equation of the tangent line to y = p(x) at $x = x_1$ *is*

$$y - (-0.875) = 5.75\left(x - \left(-\frac{3}{2}\right)\right).$$

Letting y = 0, we find: $x_2 = \frac{0.875}{5.75} - \frac{3}{2} = -1.3478260869565217 \approx \mathbf{-1.35}$

2. *[Stewart problem, 10 pts]* Find $\lim_{x\to\infty}(e^x + x)^{1/x}$

Solution: Let $y = (e^x + x)^{\frac{1}{x}}$.

The limit as $x \to \infty$ *is an indertminate form:* ∞^0.

To use L'Hôpital's rule, we convert this to an indeterminate form of type $\frac{\infty}{\infty}$.

Now, $\ln y = \ln(e^x + x)^{1/x} = \frac{1}{x}\ln(e^x + x) = \frac{\ln(e^x+x)}{x}$ *is of the form* $\frac{\infty}{\infty}$.

Invoking L'Hôpital's rule, $\lim_{x\to\infty}\frac{\ln(e^x+x)}{x} = \lim_{x\to\infty}\frac{\frac{d}{dx}\ln(e^x+x)}{\frac{d}{dx}x} =$

$$\lim_{x\to\infty}\frac{\frac{1}{(e^x+x)}(e^x+1)}{1} = \lim_{x\to\infty}\frac{e^x+1}{e^x+x}$$

Now, since this new limit is of the form $\frac{\infty}{\infty}$, *we may invoke L'Hôpital's rule once again, viz.*

$$\lim_{x\to\infty}\frac{e^x+1}{e^x+x} = \lim_{x\to\infty}\frac{\frac{d}{dx}(e^x+1)}{\frac{d}{dx}(e^x+x)} = \lim_{x\to\infty}\frac{e^x}{e^x+1} = 1$$

Now since $\ln y \to 1$ *as* $x \to \infty$, $y = e^{\ln y} \to e^1 v = e$ *as* $x \to \infty$.

3. *[Stewart problem, 10 pts]* Evaluate the indefinite integral

$$\int e^x\sqrt{1+e^x}\ dx$$

Solution: Let u = 1 + e^x^. Then du = e^x^ dx. Thus

$$\int e^x\sqrt{1+e^x}\ dx = \int \sqrt{u}\ du = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(1+e^x)^{\frac{3}{2}} + C$$

4. *[Stewart problem, 10 pts]* Evaluate the Riemann integral

$$\int_0^{\frac{1}{4}} \frac{\arcsin(2x)}{\sqrt{1-4x^2}}\,dx$$

Solution: Let $w = arcsin(2x)$. $Then\ dw = \frac{2}{\sqrt{1-4x^2}}\,dx \Rightarrow \frac{dw}{2} = \frac{1}{\sqrt{1-4x^2}}\,dx.$

Also, as x varies from $0\ to\,\frac{1}{4},\ w\ varies\ from\ 0\ to\,\frac{\pi}{6}.$ *Thus*

$$\int_{x=0}^{x=\frac{1}{4}} \frac{arcsin(2x)}{\sqrt{1-4x^2}}\,dx = \int_{w=0}^{w=\frac{\pi}{6}} \frac{arcsin(2x)}{\sqrt{1-4x^2}}\,dx =$$

$$\int_{u=0}^{u=\frac{\pi}{6}} arcsin(2x)\frac{1}{\sqrt{1-4x^2}}\,dx =$$

$$\int_{u=0}^{u=\frac{\pi}{6}} w\ \frac{1}{\sqrt{1-4x^2}}\,dx = \int_{w=0}^{w=\frac{\pi}{6}} \frac{w}{2}\,dw = \frac{w^2}{4}\Bigg[\begin{matrix} w=\frac{\pi}{6} \\ w=0 \end{matrix} \quad = \frac{\boldsymbol{\pi^2}}{\mathbf{144}}$$

Extra Credit *[Stewart problem, 10 pts*:

$$\int x^3\sqrt{1+x^2}\ \ dx$$

Solution:

Let $t = 1 + x^2.\ Then\ dt = 2x\,dx\ \Rightarrow\ x\,dx = \frac{dt}{2}.$

$Also, \quad note\ that\ x^2 = t - 1.\quad So$

$$\int x^3\sqrt{1+x^2}\ \ dx = \int x^2\sqrt{1+x^2}\ \ (x\,dx) = \int (t-1)\sqrt{t}\,\frac{dt}{2} =$$

$$\int (t-1)\sqrt{t}\,\frac{dt}{2} = \frac{1}{2}\int \left(t^{\frac{3}{2}} - t^{\frac{1}{2}}\right) dt = \frac{1}{2}\left(\frac{2}{5}t^{\frac{5}{2}} - \frac{2}{3}\,t^{\frac{3}{2}}\right) + C =$$

$$\frac{1}{5}t^{\frac{5}{2}} - \frac{1}{3}\,t^{\frac{3}{2}} + C = \boldsymbol{\frac{1}{5}(1+x^2)^{\frac{5}{2}} - \frac{1}{3}\,(1+x^2)^{\frac{3}{2}} + C}$$

But just as much as it is easy to find the differential [derivative] of a given quantity, so it is difficult to find the integral of a given differential. Moreover, sometimes we cannot say with certainty whether the integral of a given quantity can be found or not.

– Johann Bernoulli