MATH 161 SOLUTIONS: QUIZ II 8TH SEPTEMBER 2017

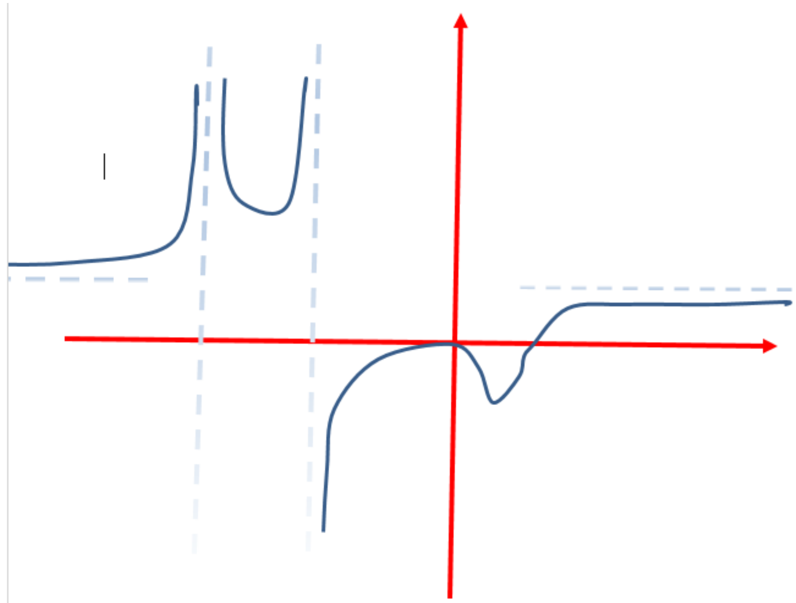
(Calculator Free)

1. *[10 pts]* The graph of a rational function is shown below. Assume that

zeroes: x = 0, x = 3

singularities: x = -2, x = -4

limiting behavior: y ⟶ 3 as |x| ⟶ ∞



Find an equation of a rational function that incorporates all of this information. (Note that this problem has more than one correct answer.)

Solution:

Given the information about the zeros, we find that x and x + 3 must be factors of the numerator.

Given the information about the singularities, x + 2 and x + 4 must be factors of the denominator.

Since the zero at x = 0 does not create a sign change, we find that x^2 or any even power of x, must be a factor of the numerator. Since the singularity at x = -4 also results in no sign change, we find that $(x + 4)^2$ or any even power of x + 4, must be a factor of the denominator.

So our first guess is:

$$y = \frac{x^2(x-3)}{(x+2)\,(x+4)^2}$$

Noting that the value of y as x ⟶ ∞ is 1, we have only to make one change:

$$y = \frac{3x^2(x-3)}{(x+2)(x+4)^2}$$

Of course, there are infinitely many other functions that would satisfy the requirements.

2. *[6 pts each]* Compute each of the following limits. *Explain your reasoning.*

(*a*) $\lim_{x\to\infty} \dfrac{(2x^3+11)^2(3x-91)^3}{(2x^2+5)^4(x+2017)}$

Solution: Observe that:

$$\frac{(2x^3+11)^2(3x-91)^3}{(2x^2+5)^4(x+2017)} \cong \frac{(2x^3)^2(3x)^3}{(2x^2)^4 x} = \frac{4(27)}{16}\left(\frac{x^9}{x^9}\right) \to \frac{27}{4} \text{ as } x\to\infty$$

(*b*) $\lim_{x\to 2} \dfrac{\frac{1}{x^2}-\frac{1}{4}}{x-2}$

Solution: Observe that, as long as x ≠ 2:

$$\frac{\frac{1}{x^2}-\frac{1}{4}}{x-2} = \frac{\frac{4}{4x^2}-\frac{x^2}{4x^2}}{x-2} = \frac{4-x^2}{4x^2(x-2)} =$$

$$\frac{-(x-2)(2+x)}{4x^2(x-2)} = \frac{-(2+x)}{4x^2} \to -\frac{4}{16} = -\frac{1}{4} \text{ as } x\to 2$$

(*c*) $\lim_{x\to 5} \dfrac{2x^2-9x-5}{x^2-8x+15}$

Solution: Observe that, as long as x ≠ 5:

$$\frac{2x^2-9x-5}{x^2-8x+15} = \frac{(x-5)(2x+1)}{(x-5)(x-3)} = \frac{2x+1}{x-3} \to \frac{11}{2} \text{ as } x\to 5$$

(*d*) $\lim_{x\to 0} \dfrac{\sqrt{x+4}-2}{x}$

Solution: We begin by rationalizing the numerator of the algebraic expression. Then we assume that, as long as x ≠ 0:

$$\frac{\sqrt{x+4}-2}{x} = \left(\frac{\sqrt{x+4}-2}{x}\right)\left(\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}\right) = \frac{x}{(x)(\sqrt{x+4}+2)} =$$

$$\frac{1}{\sqrt{x+4}+2} \to \frac{1}{\sqrt{4}+2} = \frac{1}{4} \quad \text{as } x\to 0.$$

3. *[8 pts]* Does $\lim_{x\to 1} g(x)$ exist given that

$$\text{Let } g(x) = \frac{3x^2-4x+1}{x^4-1}$$

If so, find it; if not explain! (*Hint:* Factor first.)

Solution: Let's begin by factoring, noting that the denominator is a difference of two squares.

$$g(x) = \frac{(x-1)(3x-1)}{(x^2+1)(x+1)(x-1)}$$

Now, as x →1, we can cancel the x – 1 factor occurring both in the numerator and the denominator.

So, for x ≠ 1:

$$g(x) = \frac{3x-1}{(x^2+1)(x+1)}$$

Now, as x →1, g(x) → 2/4 = ½ .

Thus $\lim_{x\to 1} g(x)$ *exists and equals ½.*

EXTRA CREDIT: *University of Michigan* calculus problem (first exam, 7 Oct 2014)

Consider the function y = h(x) defined by:

$$h(x) = \begin{cases} \dfrac{60(x^2-x)}{(x^2+1)(3-x)} & for\ x < 2 \\ 9 & for\ x = 2 \\ 5e^{ax} - 1 & for\ x > 2 \end{cases}$$

Solution: *In order for* $\lim_{x\to 2} h(x)$ *to exist, it must be true that* $\lim_{x\to 2+} h(x) = \lim_{x\to 2-} h(x)$

Now $= \lim_{x\to 2-} h(x)$ *= 60(2*^{2} *− 2) /((2*^{2} *+ 1)(3 − 2)) = 24 and* $\lim_{x\to 2+} h(x)$*= 5e* ^{2a} *− 1.*

So it follows that 5e ^{2a} *− 1 = 24.*

Solving for a, we have 5e ^{2a} *− 1 = 24;* *e* ^{2a} *= 5;* *thus a = ln(5)/2 ≈ 0.804.*