MATH 161 SOLUTIONS: QUIZ III

15 SEPTEMBER 2017

1. (a) *[4 pts] Carefully* state the *Intermediate Value Theorem.*

Theorem: Let y = f(x) be a continuous function on an interval [a, b]. Let z be any number between f(a) and f(b). Then there exists a number c in the interval [a, b] for which f(c) = z.

(b) *[5 pts]* Using the IVT explain why the function f(x) = x + 3 – 2 sin x must have *at least one real root.*

Solution: *Note that f(0) = 3 > 0 and G(□) = 3 + 3 – 2 sin x > 6 – 2 = 4 > 0*

Also note that f is continuous on the interval [0, 4]. Since G(4) < 0 < G(0), the IVT guarantees the existence of a root of the equation G(x) = 0 in the interval [0, 4].

2. *[5 pts]* Using the *Squeeze Theorem*, show that the function

$$f(x) = 3x^4 \cos\left(\frac{x+1732}{x^3}\right)$$

has a limit as x → 0 and find the value of this limit. (You need not state the general theorem; only show how it can be applied here.)

Solution: *First, a brief review of the Squeeze theorem.*

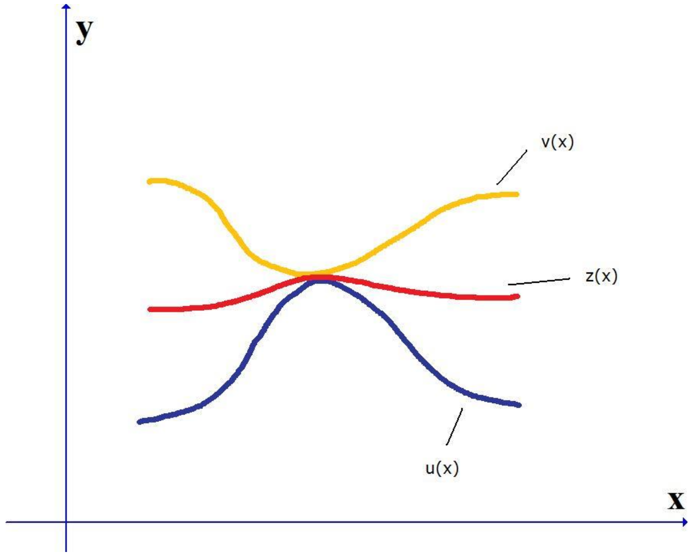
Theorem: Let v(x), z(x), u(x) be functions defined on an interval (a, b) except possibly at the point p, where p ∈(a, b)

In addition, assume that

$$\lim_{x\to p} v(x) = L \textit{ exists and } \lim_{x\to p} u(x) = L \textit{ exists}.$$

Then

$$\lim_{x\to p} z(x) = \textit{exists and, furthermore, } \lim_{x\to p} z(x) = L.$$



Returning to the original question:

Using the Squeeze Theorem, find $f(x) = x + 3x^4 \cos\left(\frac{x+1732}{x^3}\right)$.

Solution:

$$-1 \le \cos\left(\frac{x}{x^3+1732}\right) \le 1 \Rightarrow$$

$$-3x^4 \le 3x^4 \cos\left(\frac{x}{x^3+1732}\right) \le 3x^4$$

Now, as $x \to 0$, $3x^4 \to 0$ *and* $-3x^4 \to 0$.

Thus, invoking the Squeeze Theorem, we obtain:

$$\lim_{x\to 0} 3x^4 \cos\left(\frac{x}{x^3+1732}\right) = 0$$

3. *[12 pts]* Compute each of the following limits. As usual, show your work.

(A) $\lim_{x\to 0} \frac{x}{\cos 4x}$

Solution: *Since* $\lim_{x\to 0} \cos 4x \ne 0$ *we can use the limit law for quotients:*

$$\lim_{x\to 0} \frac{x}{\cos 4x} = \frac{\lim_{x\to 0} x}{\lim_{x\to 0} \cos 4x} = \frac{0}{1}$$

(B) $\lim_{x\to 0} \frac{\sin 5x}{x}$

Solution: $\lim_{x\to 0} \frac{\sin 5x}{x} = \lim_{x\to 0} 5\,\frac{\sin 5x}{5x} = 5\lim_{x\to 0} \frac{\sin 5x}{5x} = 5\lim_{t\to 0} \frac{\sin t}{t} = 5$

(C) $\lim_{x\to\frac{\pi}{2}} \frac{\sin x}{x}$

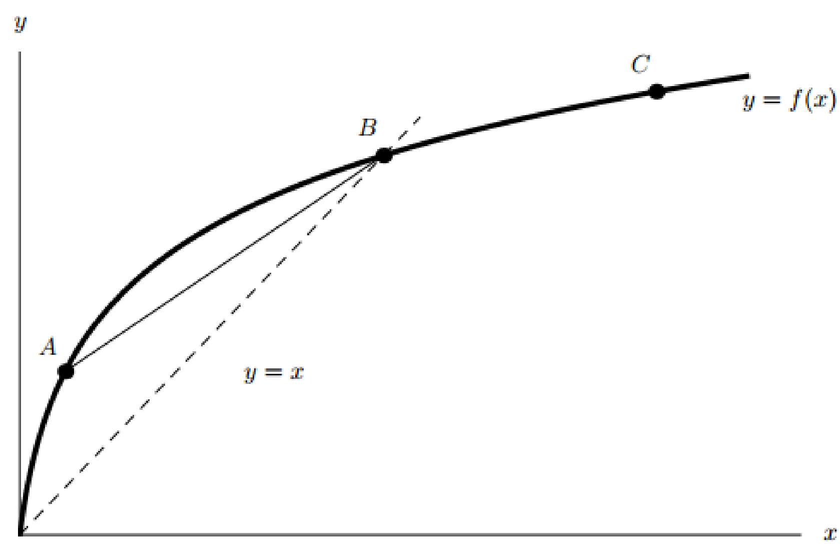
Solution: *Since* $\lim_{x\to\pi/2} x \neq 0$ we can use the limit law for quotients:

$$\lim_{x\to\frac{\pi}{2}} \frac{\sin x}{x} = \frac{\lim_{x\to\pi/2} \sin x}{\lim_{x\to\pi/2} x} = \frac{1}{\left(\frac{\pi}{2}\right)} = \frac{2}{\pi}$$

4. *[6 pts]* For the graph of y = f (x) in the figure below, arrange the following numbers from ***smallest to largest:***

A The slope of the curve at A.
B The slope of the curve at B.
C The slope of the curve at C.
AB The slope of the line *AB*.
0 The number 0.
1 The number 1.

Explain the positions of the number 0 and the number 1 in your ordering. Any unclear answers will not receive credit.



0 < **C** < **B** < **AB** < **1** < **A**

Solution: The number one and all other slopes are positive, so 0 must be the smallest number. The line $y = x$ has a slope of 1. The slope at C, the slope at B, and the slope of the line AB are each smaller than the slope of the line $y = x$ by looking at the picture. The slope at A is larger than the slope of $y = x$ also by the picture. Thus 1 is the second to largest number in the ordering.

5. *[2 pts]* To find the derivative of $g(x)=2x^2+5x-9$ at $x = 8$ algebraically, one must evaluate the following expression.

(A) $\lim_{h\to 0}\dfrac{2(8+h)^2+5(8+h)-(2\cdot 8^2+5\cdot 8-9)}{h}$

(B) $\dfrac{g(8+1)+g(8)}{h}$ (C) $\lim_{h\to\infty}\dfrac{g(h)-g(8)}{h}$

(D) All of the above are correct. (E) None of the preceding is correct

➢ *Explain how you arrived at your answer!*

Solution: *The correct answer is (E):* **None of the preceding is correct**.

Note that (A) is not correct since it is missing -9: The correct version of (A) is:

$$\lim_{h\to 0}\frac{2(8+h)^2+5(8+h)-9-(2\cdot 8^2+5\cdot 8-9)}{h}$$

6. *[2 pts]* Given the following data about a function $f(x)$, the equation of the tangent line at $x = 5$ is approximated by

x	3	3.5	4	4.5	5	5.5	6
$f(x)$	10	8	7	4	2	0	-1

(A) $y-5=-4(x-2)$ (C) $y-2=-4(x-5)$

(B) $y-5=-8(x-2)$ (D) $y-2=-8(x-5)$

➢ *Explain how you arrived at your answer!*

Solution: *Approximating the slope of the curve $y = f(x)$ at $x = 5$:*

$f'(5)\approx\frac{f(5.5)-f(5)}{5.5-5}=\frac{0-2}{0.5}=-4$ or $f'(5)\approx\frac{f(4.5)-f(5)}{4.5-5}=\frac{4-2}{-0.5}=-4$

So the slope of the tangent line to $y=f(x)$ at $x=5$ is approximately -4.

Note that $f(5)=2$.

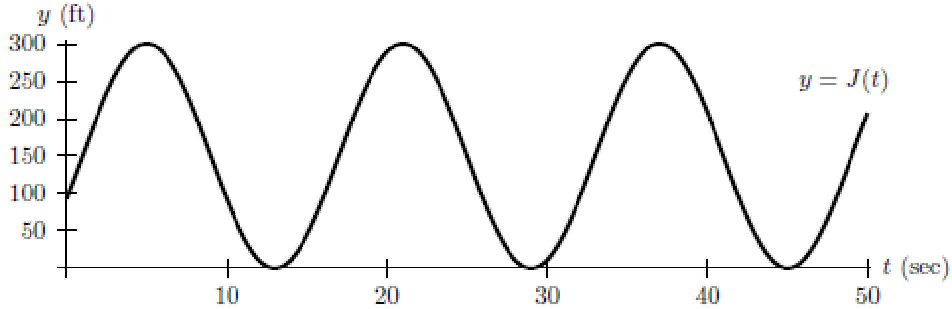
Using point-slope form: $y-2=-4(x-5)$

So the correct choice is (C).

7. [University of Michigan] Odette, a dare devil, jumps off the side of a bungee jumping platform while attached to a magically elastic bungee cord. Just a few moments after the jump begins, a timer is started and her position is recorded. At t seconds after the timer begins, her distance in feet below the platform is given by the function

$$J(t)=-150\cos(0.125\pi(t+3))+150.$$

A portion of the graph of $y = J(t)$ is shown below.



Throughout this problem, do not make estimates using the graph.

(a) *[2 pts]* Compute the *average* velocity of the bungee jumper during the first 16 seconds after the timer begins.

Solution: Since $0.125\pi = 2\pi/(\text{period of } J(t))$, the period of $J(t)$ is $2\pi/(0.125\pi) = 16$. Thus, $J(0) = J(16)$ and

$$\text{average velocity} = \frac{J(16) - J(0)}{16} = \frac{0}{16} = 0 \text{ ft/s}.$$

Answer: average velocity = 0 **ft/s**

(b) *[3 pts]* Compute the *average speed* of the bungee jumper during the first 16 seconds after the timer begins.

Hint: Recall that *average speed* over an interval of time is given by $\frac{\textit{distance traveled}}{\textit{time elapsed}}$.

Solution: Since $J(t)$ has period 16 and amplitude 150,

$$\text{distance traveled during the interval } 0 \le t \le 16 = 4(\text{amplitude}) = 600 \text{ ft}$$

Thus, the average speed is $(600 \text{ ft})/(16 \text{ s}) = 37.5$ ft/s.

Answer: average speed = 37.5 **ft/s**

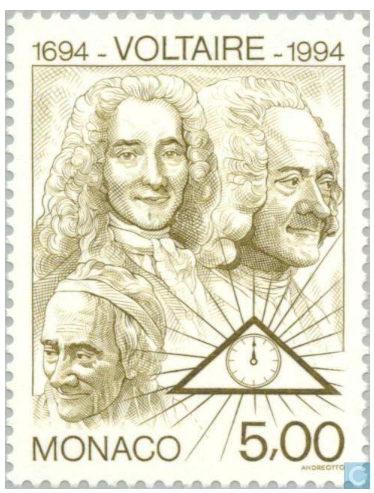
(c) *[4 pts]* Use the *limit definition* of instantaneous velocity to write an explicit expression for the instantaneous velocity of the bungee jumper 2 seconds after the timer begins. *Your answer should not involve the letter J. Do not attempt to evaluate or simplify the limit.*

Answer: $\lim_{h\to 0} \frac{-150\cos(0.125\pi(5+h)) + 150 - (-150\cos(0.125\pi(5)) + 150)}{h}$

(d) *[2 pts]* Find all values of t in the interval $0 \le t \le 30$ when the instantaneous velocity of the bungee jumper is 0 feet per second.

Solution: The instantaneous velocity is 0 when the tangent line to the position graph is horizontal. This occurs at the maxima and minima on the sinusoidal graph. The first maximum occurs at $t = 5$ and so the first minimum occurs at $t = 13$ (half a period later).

Answer: $5, 13, 21, 29$



The more you know, the less sure you are.

\- Voltaire