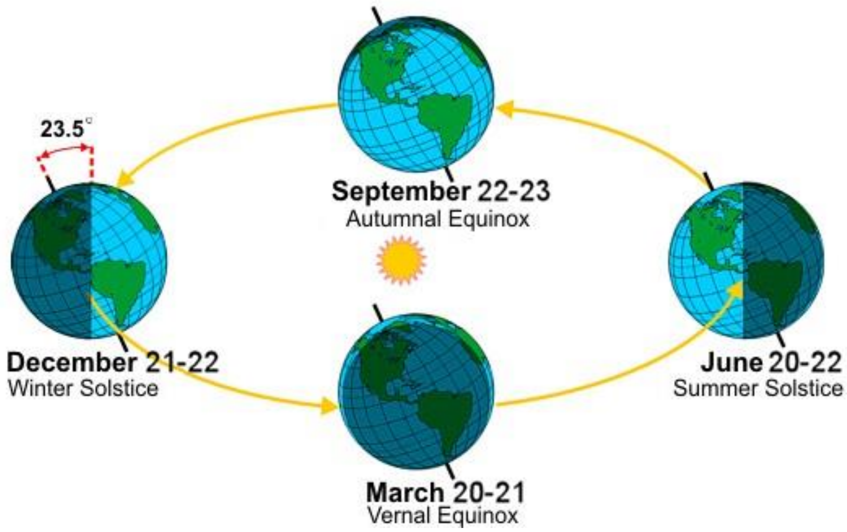
MATH 161 SOLUTIONS: QUIZ IV 22 SEPTEMBER 2017



1. *[2 pts]* Suppose $f(x) = \left(x + \frac{1}{2}\right) e^x$

Albertine discovers that the derivative of $f(x)$ *is* $f'(x) = \left(x + \frac{3}{2}\right) e^x$.

Using this formula for $f'(x)$, write an equation for the tangent line to the graph of $f(x)$ at $x = 2$.

Solution: The slope is $m = (2 + \frac{3}{2})\, e^2 = \frac{7}{2} e^2$ *and the y-coordinate when* $x = 2$ *is*

$f(2) = \left(2 + \frac{1}{2}\right) e^2 = \frac{5}{2} e^2$. *Using the point-slope form for the tangent line, we obtain:*

$$y - \frac{5e^2}{2} = \frac{7}{2} e^2 (x - 2)$$

2. *[2 pts each]* Differentiate each of the following functions. (You may use shortcuts. Simplify your answer when appropriate.)

(a) $y = x \cos x + e^e + x^{1789}$

Solution: Using the product rule,

$dy/dx = x(d/dx)\cos x + \cos x\,(d/dx)(x) + (d/dx)e^e + (d/dx)\, x^{1789} =$

$x(-\sin x) + \cos x + 0 + 1789\, x^{1788} =$

$\cos x - x \sin x + 1789\, x^{1788}$

(b) $y = (x^4 + 1)\, e^x$

Solution: Using the product rule, $dy/dx = (x^4 + 1)\,(d/dx)e^x + e^x\,(d/dx)\,(x^4 + 1) =$

$e^x (x^4 + 1) + e^x\, 4x^3 = e^x (x^4 + 4x^3 + 1)$

(c) $y = 5 \cos x - x^3 \sin x + 2017^{\pi}$

Solution: Using the product rule, $dy/dx = -5 \sin x - (x^3\,(d/dx) \sin x + \sin x\,(d/dx)\,(x^3)) + 0 =$

$-5 \sin x - (x^3 \cos x + (\sin x)(3x^2)) = -5 \sin x - x^3 \cos x - 3x^2 \sin x$

3. *[3 pts]* Find the *critical points* of the function $y = g(x) = 7x^9 - 36x^7 + 1789$

Solution: $g'(x) = 9(7)x^8 - 7(36)x^6 + 0 = 63x^6(x^2 - 4) = 63x^6(x+2)(x-2)$

$Setting\ \ g'(x) = 0\ yields\ \ \boldsymbol{x = -2, 0, 2}$

4. *[2 pts each]* Suppose that *f* and *g* are differentiable functions satisfying:

f(3) = -2, g(3) = -11, f ′(3) = 3, and g′(3) = -1.

(a) Let y = f(x) – 5g(x) + 2017. Compute $\dfrac{dy}{dx}$ at x = 3.

Solution: Using the rules for differentiating sums, differences and noting that 2017 is a constant, we have:

dy/dx = f ′(x) – 5 g′(x) + 0

And so when x = 3, dy/dx = f ′(3) – 5 g′(3) + 0 = 3 – (5)(-1) = ***8***

(b) Let $y = f(x)g(x)$. Compute $\dfrac{dy}{dx}$ at x = 3.

Solution: Using the product rule:

dy/dx = f(x) g′(x) + f ′(x) g(x)

And when x = 3, dy/dx = f(3) g′(3) + f ′(3) g(3) = (-2)(-1) + (3)(-11) = ***- 31***

5. *[4 pts]* Find an equation of the *normal line* to the curve

$$y = (1 + x + x^2)(2 - 3x + 5x^4)$$

at the point (with x-coordinate) x = 0.

Recall that a line is said to be "normal" to the curve at x = p if it is *perpendicular* to the tangent line at x = p.)

Solution: The point of tangency is P = (0, 2).

Next, dy/dx = $(1 + x + x^2)(-3 + 20x^3) + (1 + 2x)(2 - 3x + 5x^4)$

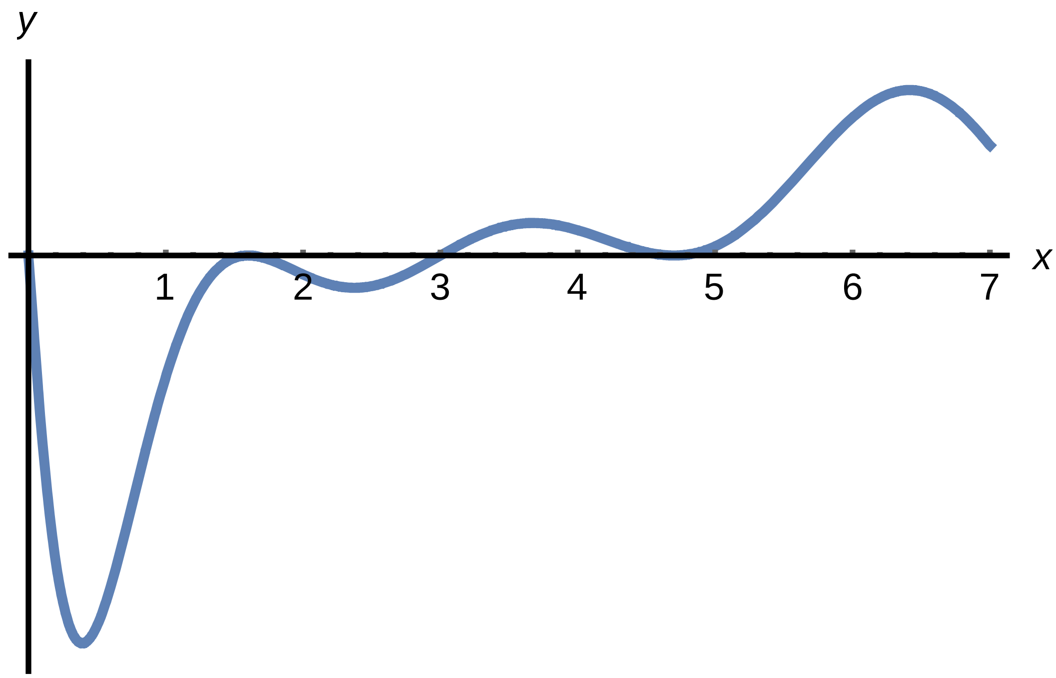
Thus, the slope of the tangent line at P is (1)(-3) + (1) (2) = -1.

So the slope of the normal line is +1.

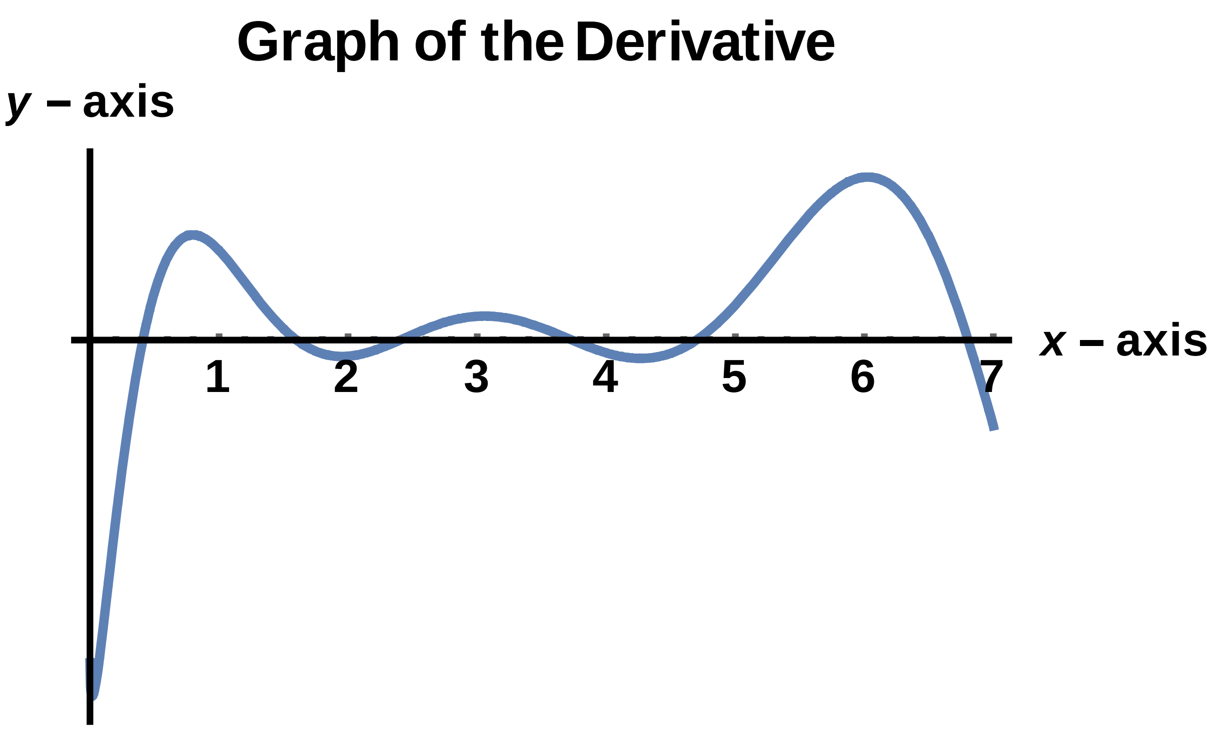
Finally, the equation of the tangent line is: y – 2 = 1(x – 0) which simplifies to ***y = x + 2****.*

6. *[5 pts]* Using the process of "geometric differentiation," sketch the graph of the derivative of the function y = G(x) whose graph is given below.

FYI: This is the graph of $y = x(x-1.6)^2(x-3)(x-4.7)^2(x-8)^2(1+\sqrt{x})e^{-x/8}$



Solution: Begin by finding the zeroes of dy/dx by looking for horizontal tangent lines in y = f(x). Then perform a sign analysis on dy/dx by looking for regions of increase and decrease in y = f(x).



7. Charlotte the spider has chosen to live on the y-axis.

She was ***born at time t = 0.*** Her *position* at time *t* (hours) is given by

$$y(t) = \frac{1}{80}\left(t^4 - 8t^3 + 10t^2 + 1\right) \text{ inches.}$$

(a) *[1 pt]* Find Charlotte's (*instantaneous) velocity* at time *t*. (You may use shortcuts for differentiation.)

Solution:

$$v(t) = \frac{dy}{dt} = \frac{1}{80}\left(4t^3 - 24t^2 + 20t\right) = \frac{1}{20}\left(t^3 - 6t^2 + 5t\right) =$$

$$\frac{1}{20}t\,(t-1)(t-5)\ \textit{inches / hour}$$

(b) *[1 pt]* When is Charlotte moving *south?* Explain.

Solution:

*Charlotte moves south when v(t) < 0. Performing a sign analysis on v(t), we find that v(t) < 0 precisely when **1 < t < 5 hours**.*

EXTRA CREDIT

Marcel, a Math 161 student, realizes that the more caffeine he consumes, the faster he completes his WebAssign homework. Before starting tonight's assignment, he buys a cup of coffee containing a total of 100 milligrams of caffeine.

Let $T(c)$ be the number of minutes it will take Marcel to complete tonight's assignment if he consumes c milligrams of caffeine. Suppose that T is continuous and differentiable.

a) *[1 pt]* Circle the one sentence below that is best supported by the statement

"the more caffeine Marcel consumes, the faster he completes his online homework assignments."

i. $T'(c) \geq 0$ for every value c in the domain of T.

ii. $T'(c) \leq 0$ for every value c in the domain of T.

iii. $T'(c) = 0$ for every value c in the domain of T.

Solution: Sentence ii, since more caffeine results in faster performance.

b) *[2 pts]* Explain, in the context of this problem, why it is reasonable to assume that $T(c)$ has an inverse. (In other words, T^{-1} exists.)

Solution: Note that the more caffeine that Marcel consumes, the faster he will succeed in completing his assignment. So T(c) is a strictly decreasing function, and hence has an inverse.

c) *[2 pts]* Interpret the equation $T^{-1}(100) = 45$ in the context of this problem. Use a complete sentence and include units.

Solution: In order for Marcel to complete his WebAssign homework in 100 minutes, he must consume 45 milligrams of caffeine.

d) *[1 pt]* Suppose that p and k are constants. In the equation $T^{-1}(p) = k$, what are the units of p and k?

Corrected version: $T'(s) = v$

Answers: The units of p are minutes; the units of k are milligrams.

Answers to corrected version: The units of s are milligrams; the units of v are minutes per milligram.

e) *[2 pts]* Which of the statements below is best supported by the equation $(T^{-1})^0(20) = -10$? *Circle the* one *best answer.*

Corrected version: $(T^{-1})'(20) = -10$

i. If Marcel has consumed 20 milligrams of caffeine, then consuming an additional milligram of caffeine will save him about 10 minutes on tonight's assignment.

ii. The amount of caffeine that will result in Marcel finishing his homework in 21 minutes is approximately 10 milligrams greater than the amount of caffeine that Marcel will need in order to finish his homework in 20 minutes.

iii. The rate at which Marcel is consuming caffeine 20 minutes into his homework assignment is decreasing by 10 milligrams per minute.

iv. In order to complete tonight's assignment in 19 rather than 20 minutes, Marcel needs to consume about 10 milligrams of additional caffeine.

v. If Marcel consumes 20 milligrams of caffeine, then he will finish tonight's assignment approximately 10 minutes faster than if he consumes no caffeine.

Solution to corrected version: Sentence iv is the correct answer.

The only time my education was interrupted was when I was in school.

- George Bernard Shaw

DERIVATIVE RULES

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

$$\frac{d}{dx}\left(a^x\right) = \ln a \cdot a^x$$

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = f(x)\cdot g'(x) + g(x)\cdot f'(x)$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{\left(g(x)\right)^2}$$

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x))\cdot g'(x)$$

$$\frac{d}{dx}\left(\sin x\right) = \cos x$$

$$\frac{d}{dx}\left(\cos x\right) = -\sin x$$

$$\frac{d}{dx}\left(\tan x\right) = \sec^2 x$$

$$\frac{d}{dx}\left(\cot x\right) = -\csc^2 x$$

$$\frac{d}{dx}\left(\sec x\right) = \sec x \tan x$$

$$\frac{d}{dx}\left(\csc x\right) = -\csc x \cot x$$

$$\frac{d}{dx}\left(\arcsin x\right) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\arctan x\right) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\operatorname{arc\,sec} x\right) = \frac{1}{x\sqrt{x^2-1}}$$