MATH 161 SOLUTIONS: QUIZ VI

FRIDAY, 13TH OCTOBER 2017

Instructions: Answer any 4 of the following 5 problems. You may answer all 5 to earn extra credit.

1. (a) *[5 pts]* Suppose that the side of a huge cube of ice is *decreasing* at a rate of 0.5 cm/min when the side has length 13 cm. At what rate is the *surface area* of the cube changing at that moment?

Solution:

Let *x = side of cube (in cm). Let S = surface area of the cube (in* cm^2*).*

Given*: dx/dt = -0.5 when x = 13.*

Find*: dS/dt when r = 8*

Since S = f(x) and x = g(t), we can invoke the Chain Rule:

$$\frac{dS}{dt} = \frac{dS}{dx}\,\frac{dx}{dt}$$

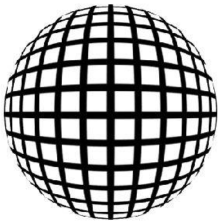
Since a cube has 6 faces, $S = 6x^2$.
Then dS/dx = 12x.

Substituting in the Chain Rule above:

$$\frac{dS}{dt} = (12x)\,(-0.5)$$

When x = 13,

$$\frac{dS}{dt} = 12(13)(-0.5) \;=\; \boldsymbol{-78\ cm^2/min}$$



(b) *[5 pts]* Suppose that a balloon (modeled as a sphere) of radius *r* is being inflated and that, at the moment when r = 8 cm, its radius is changing at the rate of 3 cm/sec. How quickly is the *surface area* of the balloon changing at the moment when the balloon's radius is 8 cm? (*Hint:* The surface area of a sphere of radius *r* is given by $S = 4\pi r^2$.)

Solution:

Let *r denote the radius of the sphere (in cm).*

Let *S be the surface area of the sphere (in cm^2).*

Given*:* $dr/dt = 3$ *when* $r = 8$.

Find*:* dS/dt *when* $r = 8$

Since $S = f(r)$ *and* $r = g(t)$, *we can invoke the Chain Rule:*

$$\frac{dS}{dt} = \frac{dS}{dr}\frac{dr}{dt}$$

Now: $dS/dr = 8\pi r$; *evaluating at* $r = 8$: $dS/dr = 64\pi$.

Since $dr/dt = 3$, *we have:*

$$dS/dt = (64\pi)\,3 = \mathbf{192\pi\ cm^2/sec} \approx \mathbf{603.2\ cm^2/sec}$$

2. *[10 pts]* Harry the potter has a *fixed* volume of clay in the form of a cylinder. As he rolls the clay, the length of the cylinder, *L*, increases, while the radius, *r*, decreases. If the length of the cylinder is increasing at a constant rate of 0.2 cm per second, find the rate at which the radius is changing when the radius is 1.5 cm and the length is 4 cm. *(You need not simplify your numerical answer.)*

Solution:

We are given that $dV/dt = 0$ *(since the potter has a fixed volume of clay).*

Also, we are given that $dh/dt = 0.2$.

We know that $V = \pi r^2 h$. *Since each of r and h is a function of time, we may differentiate implicitly to obtain:*

$$0 = dV/dt = \pi\{ r^2 (dh/dt) + 2rh\,(dr/dt)\}.$$

Thus $0 = 0.2\, r^2 + 2rh\, (dr/dt)$.

When $r = 1.5$ and $h = 4$, we find that:

$$0 = 0.2(1.5)^2 + 2(1.5)(4)\,(dr/dt)$$

Hence $dr/dt = -1.5/8 =$ ***-0.0375 cm/sec***

3. Albertine and Jean-Luc were friends in high school but then went to college in different parts of the country. They thought they were going to see each other in Detroit during December break, but their schedules didn't match up. In fact, it turns out that Albertine is leaving on the same day that Jean-Luc is arriving.
Shortly before Jean-Luc's train arrives in Detroit, he sends a text to Albertine to see where she is, and Albertine sends a text response to say that, sadly, her train has already left. At the moment Albertine sends her text, she is 20 miles due east of the center of the train station and moving east at 30 mph while Jean-Luc is 10 miles due south of the train station and moving north at 50 mph.

(a) *[3 pts]* What is the distance between Jean-Luc and Albertine at the time Albertine sends her text? *Remember to include units.*

Solution:

Let *x, y, z be the distances (in miles) between Albertine and the station, Swann and the station, and Albertine and Swann, respectively.*

Then, since the triangle is a right triangle, $x^2 + y^2 = z^2$.

When Albertine sends her text, $x = 20$ and $y = 10$. So Albertine and Swann are

$$\sqrt{20^2+10^2} = 10\sqrt{5} \approx \mathbf{22.4\ miles\ apart}.$$

(b) *[7 pts]* When Albertine sends her text, are she and Jean-Luc moving closer together or farther apart? How quickly? *(You need not simplify your numerical answer.) You must show your work clearly to earn any credit. Remember to include units.*

Solution:

Differentiating $x^2 + y^2 = z^2$ implicitly (with respect to t) yields:

$2x\,(dx/dt) + 2y\,(dy/dt) = 2z(dz/dt)$

When Swann sends his text, we know that $dx/dt = 30$ and $dy/dt = -50$. Hence

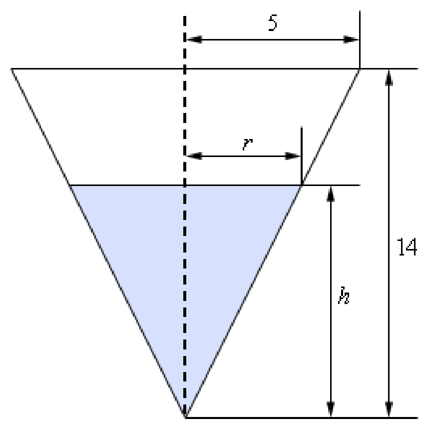
$$\frac{dz}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{z} = \frac{(20)(30)+(10)(-50)}{10\sqrt{5}} = 2\sqrt{5} \approx 4.47$$

Since, at this time, the sign of dz/dt is positive, Albertine and Swann are moving closer to each other at a rate of approximately ***4.47 miles/hr.***

4. *[10 pts]* A tank of toxic chemicals in the shape of a cone is leaking the toxic liquid at a constant rate of 2 ft^3/hour. The base radius of the tank is 5 ft and the height of the tank is 14 ft.

(Recall that the volume of a cone of height H and radius R is given by: $V= (\pi/3)R^2H$.)

At what rate is the *depth* of the toxic fluid in the tank changing when the depth of the fluid is 6 ft? *(You need not simplify your numerical answer.)*



Solution:

Let *$r = r(t)$ = radius (in feet) of the circular top of the toxic fluid cone at time t and*

let *$h = h(t)$ be the height (in feet) of the fluid level at time t.*

Given: *$dV/dt = -2$ ft^3/hour.*

Find: *dh/dt when $h = 5$ ft.*

Using similar triangles, we see that $r/h = 5/14$.

Now, since $r = 5h/14$, we see that $V = (1/3)\pi r^2 h = (1/3)\pi (5h/14)^2 h =$

$(\pi/3)(5/14)^2 h^3$.

So dV/dh = $\pi(5/14)^2h^2$.

Hence dV/dt = (dV/dh) (dh/dt) = $\pi(5/14)^2 h^2$ *(dh/dt).*

We are given that dV/dt = -2. When h = 6, we have:

$$-2 = \pi(5/14)^2 6^2 \, (dh/dt)$$

Finally $dh/dt = -2(196)/\{\pi (25)(36)\} = -\frac{98}{225\pi} \approx -\mathbf{0.139}\ \frac{ft}{hour}$.

Or, we can say that the depth of the water is ***decreasing at the rate of*** $0.139\ \frac{ft}{hour}$

5. *[10 pts]* A bat moves along the curve $\mathbf{y = x^{5/2}}$ in the first quadrant in such a way that its *distance from the origin* increases at the rate of 4 cm/ sec. Find dx/dt when x = 4. *(You need not simplify your numerical answer.)*

Solution:

Let x = x(t) be the x-coordinate (in cm) of the bat at time t.

Then $y = y(t) = x^{5/2}$ *is the y-coordinate (in cm) of the bat at time t.*

Let s = s(t) be the distance (in cm) from the bat to the origin.

We are given that ds/dt = 4. Now $s^2 = x^2 + y^2 = x^2 + (x^{5/2})^2 = x^2 + x^5$.

Differentiating implicitly with respect to time:

$$d/dt\,(s^2) = d/dt\,(x^2 + x^5)$$

Hence $2s\, ds/dt = 2x\, dx/dt + 5x^4\, dx/dt = (2x + 5x^4)\, dx/dt$

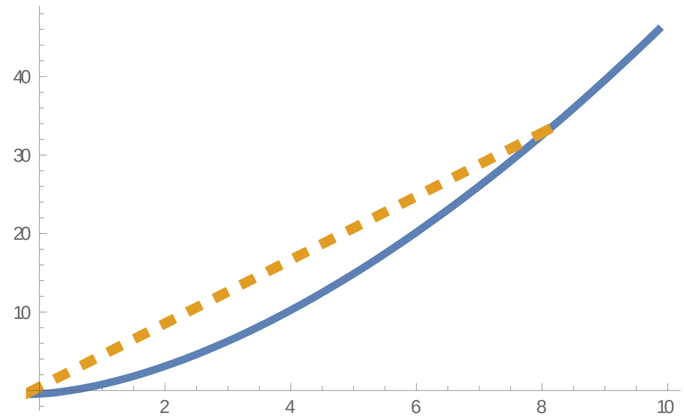
Since ds/dt = 4 and x = 4, we have 2s(4) = 1288 dx/dt.

So s = 161 dx/dt.

Now, thanks to Pythagoras, when x = 4, $s^2 = 4^2 + 4^5 = 4^2(1 + 4^3) = 4^2(65)$.

And so $s = 4\sqrt{65}$.

Finally, $dx/dt = s/161 = \frac{\sqrt{65}}{4}\ cm/sec$. *This is approximately* ***0.2016 cm/sec***.



The only way to learn mathematics is to do mathematics. That tenet is the foundation of the do-it-yourself, Socratic, or Texas method, ...

- Paul Halmos