MATH 161 SOLUTIONS: QUIZ VII

27 OCTOBER 2017

I hear and I forget.
I see and I remember.
I do and I understand.

- Chinese Proverb.

1. [20 pts] Find two *non-negative* real numbers whose sum is 13 such that the product of one number and the square of the other number is a *maximum*.

Reminder: Introduce your variables; state what is given; state your objective. *Find an equation of one-variable which must be maximized.* Also find the *constraints* placed upon your variable. Do not omit steps. Express answers to the nearest tenth.

Solution:

We are given the constraints: $x > 0$, $y > 0$, *and* $x + y = 13$.

Our goal is to maximize the product, $P = xy^2$.

Also note that $13 > x > 0$, *and* $13 > y > 0$ *(Why?)*

Solving for y in terms of x: $y = 13 - x$ *and then substituting into the formula for P yields:*

$$P = x(13 - x)^2 \text{ with domain } (0, 13).$$

Since

$dP/dx = (13 - x)^2 + x(-2)(13 - x) =$

$(13 - x)(13 - x - 2x) =$

$(13 - x)(13 - 3x)$,

the critical point is $x = 13/3$.

(Note that $x = 13$ *is not a critical point as it is an endpoint.)*

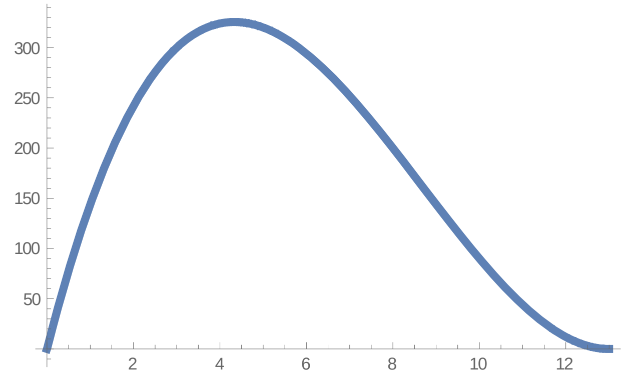
Performing a sign analysis on dP/dx, we see that,

P is rising on $(0, 13/3)$ *and falling on* $(13/3, 13)$.

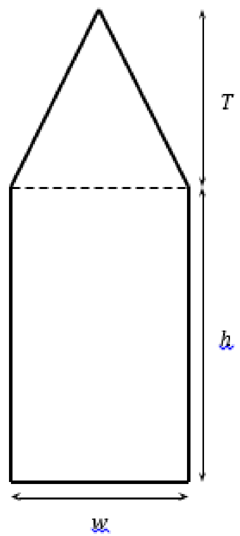
Thus P achieves a global maximum at $x = 13/3$; *now* $y = 13 - x = 26/3$.

So P is maximized when one number is ***13/3*** *and the other is* ***26/3***.

FYI here is the graph of $y = P(x)$:



2. [20 pts] Consider a window the shape of which is a rectangle of height *h* surmounted by a triangle having height *T* that is **three times** the width *w* of the rectangle (see the figure below which is not drawn to scale). If the *total* area of the window is **8 square feet**, determine the dimensions of the window which minimize the perimeter.



Reminder: Introduce your variables; state what is given; state your objective. *Find an equation of one-variable which must be minimized.* Also find the *constraints* placed upon your variable. Do not omit steps. Sketch the graph of the function of one variable that you must then minimize. Use *appropriate units.* Express answers to the nearest tenth.

Solution:

Let w = width, T = altitude of the triangle and h = height of the window, each in feet, as shown above.

Let A = area of the window

Given: *A = 8 and T = 3w.*

Goal: *Minimize perimeter, P.*

$Area = wh + \frac{1}{2} wT = wh + (3/2)\, w^2$

So $wh + (3/2)\, w^2 = 8$ (**)

Let x be the common side of the isosceles triangle.

Then $T^2 + (w/2)^2 = x^2$.

Since T = 3w,

$x^2 = 9w^2 + w^2/4 = (37/4)\, w^2$ and so $x = \frac{w\sqrt{37}}{2}$

The perimeter, $P(w, h) = w + 2h + w\sqrt{37}$

Solving for h in the Area equation (**) we obtain:

$$h = \frac{8 - \frac{3}{2}w^2}{w} = \frac{8}{w} - \frac{3}{2}w$$

Constraint on w:

Since $h \geq 0$, we have: $\frac{8}{w} - \frac{3}{2}w > 0 \Rightarrow w^2 < \frac{16}{3} \Rightarrow 0 < w < \frac{4}{3}\sqrt{3} \approx 2.31$

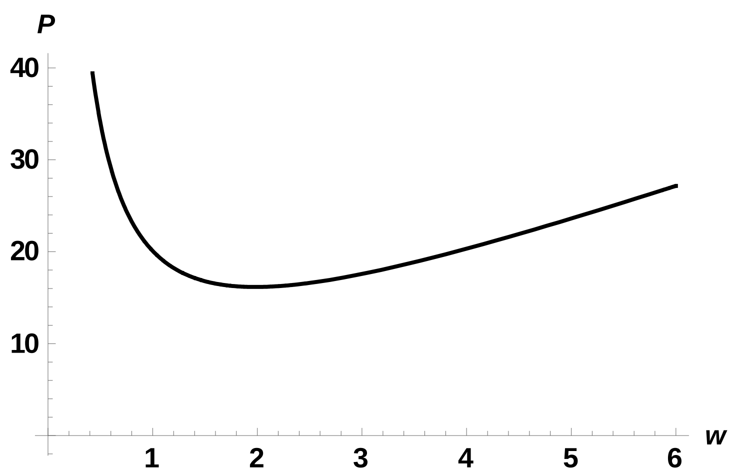
So what remains is to find the global min of the curve

$$\boldsymbol{P(w) = w + 2h + w\sqrt{37}}$$

$$= w + 2\left(\frac{8}{w} - \frac{3w}{2}\right) + w\sqrt{37} = \frac{16}{w} + w\left(\sqrt{37} - 2\right)$$

subject to the constraint $1.29 < w < 2.31\ feet$

Here is the graph of P = P(w) graphed on a larger domain (for clarity).

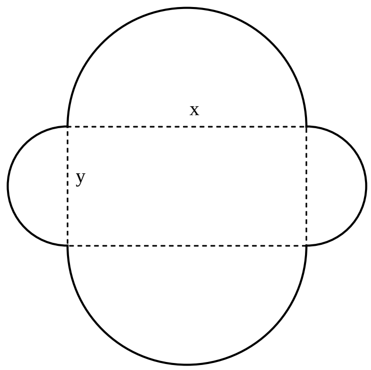


Next we anticipate one critical point, which, if such is the case, would be the global minimum.

$$\frac{dP}{dw} = -\frac{16}{w^2} + \left(\sqrt{37} - 2\right)w$$

Setting $\frac{dP}{dw} = 0$, *we find* $\boldsymbol{w} = \sqrt[3]{\frac{16}{\sqrt{37}-2}} \approx \mathbf{1.58\ feet}$

EXTRA CREDIT: *[University of Michigan]* The figure below is comprised of a ***rectangle and four semi-circles***. Units of length are given in meters.



(a) *[2 pts]* Find a formula for the enclosed area, *A*, of the figure in terms of *x* and *y*.

Solution: $A = xy + \pi(x/2)^2 + \pi(y/2)^2 =$

$\boldsymbol{xy + (\pi/4)(x^2 + y^2)}$

(b) *[2 pts]* Find a formula for the perimeter, *P*, of the figure in terms of *x* and *y*. (*Note* The perimeter does *not* include the dashed lines.)

Solution: $P = \pi x + \pi y = \boldsymbol{\pi(x + y)}$

(c) *[10 pts]* Find the values of *x* and *y* which will *maximize* the area if the perimeter is 200 meters.

Solution: Solving for y in part (b): $y = P/\pi - x = 200/\pi - x$

Substituting this expression for x in part (a):

$$A = (200/\pi - x)\,x + (\pi/4)\,x^2 + (\pi/4)\,(200/\pi - x)^2$$
$$= (200/\pi)\,x - x^2 + (\pi/4)x^2 + (\pi/4)\,(200/\pi - x)^2$$
$$= (\pi/2 - 1)\,x^2 + (200/\pi - 100)x + 10000/\pi$$

Note that the domain of A is given by $0 \le x \le 200/\pi$

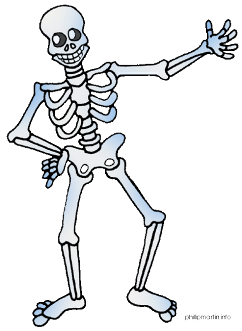
Computing dA/dx:

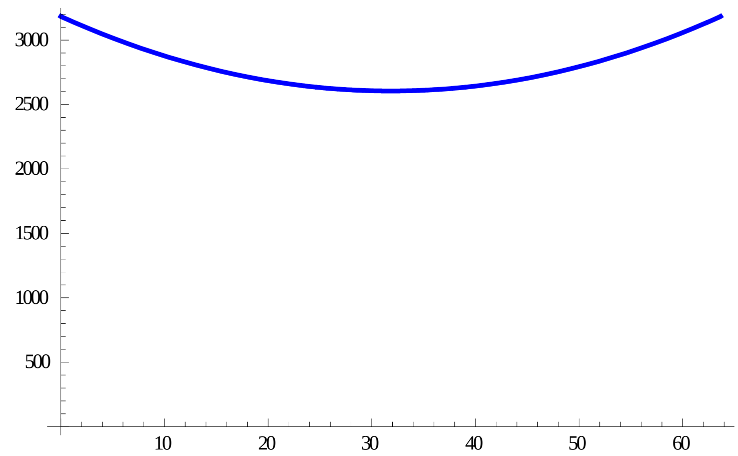
$$dA/dx = (\pi - 2)\,x + (200/\pi - 100) =$$
$$(\pi - 2)\,x + 100/\pi\,(2 - \pi) =$$
$$(\pi - 2)(x - 100/\pi)$$

The critical point of A is $x = 100/\pi$. *Note that when* $x < 100/\pi$, $dA/dx < 0$ *and when* $x > 100/\pi$, $dA/dx > 0$. *Hence A achieves a local* ***minimum*** *at the critical point. Hence the maximum of A must be achieved at an end point. Because of symmetry, the value of A at each end point is the same.*

Thus the maximum value of A is achieved when the figure reduces to a circle.

Here is a graph of A as a function of x.





Happy Halloween!