MATH 161 SOLUTIONS: TEST III 18 NOVEMBER 2017

Instructions: Answer any 9 of the following 12 problems. You may answer more than 9 to earn extra credit.

"Obviously this pardon is a forgery. But he taught himself to type, fashioned a presidential seal, put it in the mail... You sure you want to go through with this?"

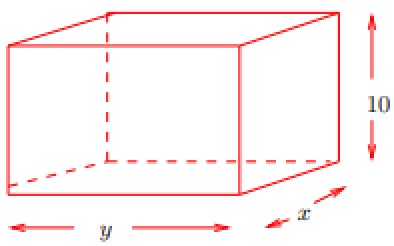
Thus mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true.

– Bertrand Russell

1. *[University of Michigan]* No matter what is done with the other exhibits, the octopus tank at the Lincoln Park Zoo must be rebuilt. (The current tank has safety issues, and there are fears that the giant octopus might escape!) The new tank will be 10 feet high and box-shaped. It will have a front made out of glass. The back, floor, and two sides will be made out of concrete, and there will be no top. The tank must contain at least 1000 cubic feet of water. If concrete walls cost $2 per square foot and glass costs $10 per square foot, use calculus to find the dimensions and cost of the least-expensive new tank. *[Be sure to show all work.]*

GIANT OCTOPUS (Enteroctopus)

Solution:



Let *x* be the width of the tank, and let *y* be the length of the tank (both in feet).

The height of the tank is given as 10 feet.

We know that the volume of the tank must be at least 1000 cubic feet, so let V denote the desired volume of the tank, where $V \geq 1000$.

Then, for a fixed value of V, we know that 10xy = V, so that x and y are related by the equation $x = \frac{V}{10y}$. Now assuming that one of the $y \times 10$ sides is the front of the tank (i.e., the glass panel), the total cost of the tank is given by:

$$C = 10(10y) + 2[(2)10x + 10y + xy] = 120y + 40x + 2xy.$$

Substituting for x, we can write C as a function of one variable:

$$C(y) = 120y + \frac{4V}{y} + \frac{V}{5}.$$

Since the cost function increases as V increases, in order to minimize the cost to build the tank we must have V be as small as possible, so we set V = 1000.

Our cost equation is now:

$$C(y) = 120y + \frac{4000}{y} + 200.$$

Taking the derivative:

$$\frac{dC}{dy} = 120 - \frac{4000}{y^2}$$

and setting the derivative equal to 0:

$$120 - \frac{4000}{y^2} = 0$$

We find that the cost function has a unique positive critical point at

$$y = \sqrt{\frac{100}{3}} \approx 5.774\ feet$$

Since the second derivative $\frac{d^2C}{dy^2} = \frac{8000}{y^3} > 0\ for\ all\ y > 0, we\ know$ that C(y) is concave up on (0, ∞). By the second-derivative test, our unique critical point must be a local (and in this case, global) minimum.

Solving for x, we find x ≈ 17.321 feet.

Thus, the glass side is the small side, and the dimensions and cost are:

Dimensions: 5.774 × 17.321 x **10 feet**

Minimum Cost: ≈ $ 1585.7

2. *[University of Michigan]* It is estimated that the rate at which people in the land of Oz will visit a new theme park is given by

$$r(t) = \frac{A}{1 + Be^{-0.5t}}$$

where A and B are both constants and r(t) is measured in people/day, and t = 0 corresponds to opening day.

(a) Write an integral that gives the total number of people visiting the park in the first year it is open. Do not try to evaluate the integral!

Solution:

The total number of people visiting the park in the first year is $\int_0^{365} r(t)dt$

(b) Suppose that A = 100 and B = 5. Given that

$$\frac{d}{dt}\left(2A\ln(1+Be^{-0.5t}) - 2A\ln(Be^{-0.5t})\right) = \frac{A}{1+Be^{-0.5t}},$$

use the Fundamental Theorem of Calculus to evaluate how many people visit the park during the first year it is open. Make sure you clearly indicate your use of the theorem.

Solution:

Let R(t) = 200 ln $(1 + Be^{-0.5t}) - 200\, ln(Be^{-0.5t})$*. Then R′(t) = r(t), so the First Fundamental Theorem of Calculus says that*

365 0 r(t) dt = R(365) − R(0) = 36, 141. $\int_0^{365} r(t)dt = R(365) - R(0) = 36{,}141.$

So 36,141 people visit the theme park during its first year.

3. *(a)* (MIT) Express the area A between the two curves as a (single) Riemann integral. *Sketch!* Do not evaluate.

$$y = 1 + x^2, \quad y = 3 + x$$

Solution:

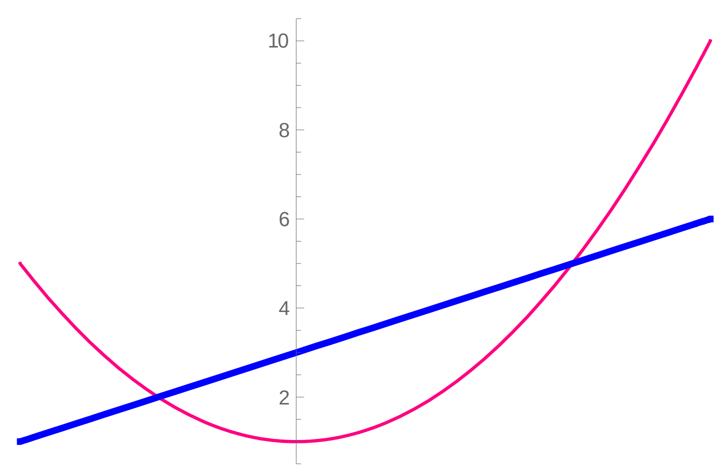
Solving for points of intersection:

$$1 + x^2 = 3 + x$$

$$x^2 - x - 2 = 0$$

$$(x - 2)(x + 1) = 0$$

$$x = -1, 2$$



Hence the area between the two curves is represented by the Riemann integral:

$$\int_{-1}^{2}(3+x-(1+x^2))\,dx$$
$$=\int_{-1}^{2}(2+x-x^2)\,dx$$

(b) Compute the *average value* of the function $f(x)=\frac{(lnx)^3}{x}$ over the interval [1, e^2]. *Simplify your answer. Sketch.*

Solution:

$$The\ \ average\ \ value\ \ of\ \ f\ \ over\ [1,\ e^2]\ \ equals$$

$$\frac{1}{e^2-1}\int_1^{e^2}\frac{(\ln x)^3}{x}\ dx=\frac{1}{e^2-1}\int_1^{e^2}(\ln x)^3(1/x)\ dx=$$

$$\frac{1}{e^2-1}\frac{(\ln x)^4}{4}\Bigg|_1^{e^2}=\frac{1}{4(e^2-1)}\left((\ln(e^2))^4-(\ln(1))^4\right)=\frac{1}{4(e^2-1)}\left((2\ln e)^4-0\right)=$$

$$\frac{1}{4(e^2-1)}\left((2)^4-0\right)=\frac{4}{e^2-1}$$

4. *[University of Michigan]* The following two parts are independent.

(a) *Suppose* $\int_{-1}^{1}g(x)\ dx=7,\ \int_{-1}^{1}(3f(x)-4g(x)+x)\ dx=2$.

Find $\int_{-1}^{1}f(x)\ dx$

Solution:

(b) Let $F(x)=\int_0^x e^{-t^2}\,dt$

(i) Find $F'(1)$

Solution: Using the FTC, version 2:

$$F'(x) = e^{-x^2}$$

So

$$F'(x) = \frac{1}{e}$$

(ii) Find $F''(1)$

Solution:

$$F''(x) = \frac{d}{dx}\,F'(x) = \frac{d}{dx}\,e^{-x^2} = -2xe^{-x^2}$$

So $F''(1) = -\frac{2}{e}$

5. (a) Verify that $f(x) = \frac{x}{x+2}$ satisfies the hypotheses of the Mean Value Theorem on the interval [1, 4] and then find all of the values, c, that satisfy the conclusion of the theorem.

Solution:

(b) Let f(x) = tan x. Show that f(π) = f(2π) = 0 but there is no number c, where π < c < 2π such that f ′(c)=0. Why does this not contradict Rolle's theorem?

Solution:

This does not contradict Rolle's theorem because f(x) is not defined at 3π/2; so of course f is not continuous on [π, 2π].

(c) $Let\ f(x) = (x-3)^{-2}$ Show that there is no value c, where 1 < c < 4, such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}$$

Why is this not a contradiction of the Mean Value Theorem?

Solution:

This does not contradict the MVT because f is not defined at x = 3, and so, of course, f is not continuous on [1, 4].

6. (a) Solve the initial value problem:

$$\frac{dy}{dt} = \sin 4t + \frac{1}{cos^2 t}$$

subject to the condition: y(π/4) = 5.

Solution:

(b) Find the **exact** value of c such that

$$\int_0^c x\sqrt{x}\,dx = \frac{4}{5}$$

Solution:

7. (a) Find $\frac{d}{dx} \int_3^x (\ln(t^5 + t + 9) - \sin(\cos t) + 1\,)\,dt$

Solution: Using the FTC, second version:

$$\frac{d}{dx}\int_3^x (\ln(t^5 + t + 9) - \sin(\cos t) + 1)\,dt \; \ln(x^5 + x + 9) - \sin(\cos x) + 1)$$

(b) Evaluate $\int_0^5 |7 - 2x|\,dx$

Solution: Splitting the region into two triangles:

$$\int_0^5 |7 - 2x|\,dx = \frac{1}{2}\left(\frac{7}{2}\right)(7) + \frac{1}{2}\left(\frac{3}{2}\right)(3) = \frac{49}{4} + \frac{9}{4} = \frac{58}{4}$$

(c) Evaluate $\int_0^5 \left(1 + 2\sqrt{25 - x^2}\right) dx$

Solution:

$$\int_0^5 \left(1 + 2\sqrt{25 - x^2}\right) dx = \int_0^5 1\,dx + 2\int_0^5 \sqrt{25 - x^2}\,dx =$$

$$5 + 2\left(\frac{1}{4}\right)\pi(5)^2$$

(d) Evaluate $\int_{-4}^4 \left(3 + \frac{4+3sin^4 x}{x^5\cos^2 x}\right) dx$

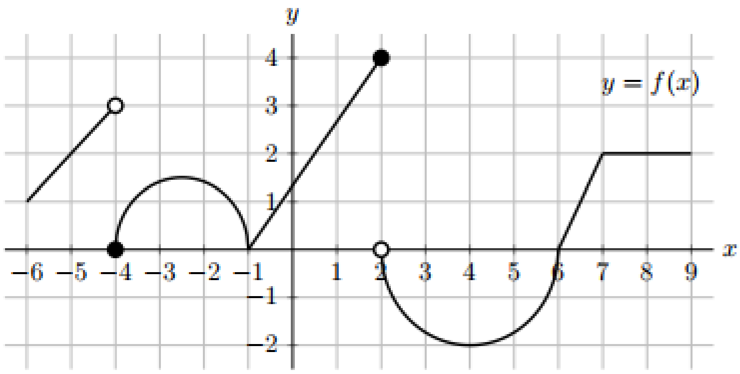
Solution: Using properties of the Riemann integral:

$$\int_{-4}^4 \left(3 + \frac{4+3sin^4 x}{x^5\cos^2 x}\right) dx = \int_{-4}^4 3\,dx + \int_{-4}^4 \frac{4+3sin^4 x}{x^5\cos^2 x}\,dx = \int_{-4}^4 3\,dx =$$

$$3\int_{-4}^4 1\,dx = 3(8) = 24$$

Since $f(x) = \frac{4+3sin^4 x}{x^5\cos^2 x}$ is an *odd* function and we are integrating over [-4, 4].

8. *[University of Michigan]* The graph of f(x) shown below consists of lines and semicircles.



Use the graph above to calculate the answers to the following questions. Give your answers as ***exact values***.
If any of the answers can't be found with the information given, write "NEI"

(a) Find the average value of f(x) on [−4, 2].

Solution: $\frac{1}{6}\int_{-4}^{2} f(x)dx = \frac{1}{6}(\frac{1}{2}\pi(1.5)^2 + \frac{1}{2}(4)(3)) = \frac{1}{6}(\frac{9\pi}{8}+6) = \frac{9\pi}{48}+1$

(b) Find the value of

$$\int_4^9 |f(z)|dz.$$

Solution: $\int_4^9 |f(z)|dz = -\int_4^6 f(z)dz + \int_6^9 f(z)dz = \frac{1}{4}\pi(2)^2 + \frac{1}{2}(3+2)(2) = 5+\pi$

(c) Find the value of T, where $4 < T \leq 9$, such that

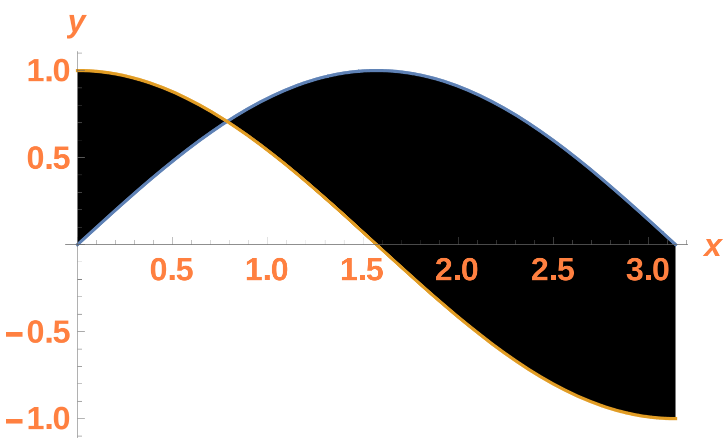
$$\int_4^T f(x)dx = 0.$$

Solution: We need to find a value of T for which

$$\int_4^T f(x)dx = \int_4^6 f(x)dx + \int_6^T f(x)dx = 0.$$

From the graph $\int_4^6 f(x)dx = -\pi$ and $\int_6^T f(x)dx = \frac{1}{2}((T-6)+(T-7))(2) = 2T-13$.
Solving for T on $2T - 13 = \pi$, we get $T = \frac{\pi+13}{2}$.

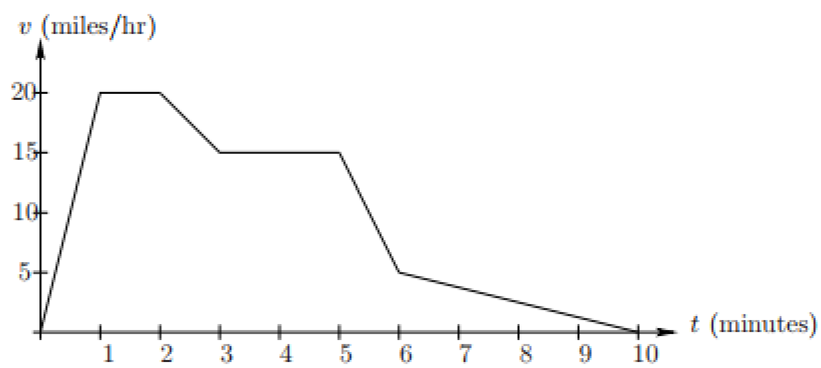
9. Express the area between y = sin x and y = cos x from 0 to π as a Riemann integral or a sum of two Riemann integrals (without using absolute value). *Do not evaluate!*
For your convenience a graph is given below. Use ***exact*** values, not guesses from the graph.



Solution: Solving sin x = cos x over [0, π] we find tan x = 1. Hence x = π/4. Now the area is given by:

$$\int_0^{\frac{\pi}{4}} (\cos x - \sin x)\ dx + \int_0^{\frac{\pi}{4}} (\sin x - \cos x)\ dx$$

10. *[University of Michigan]* Three happy wizards leave a magic show and return home to watch the Simpsons. In this episode, Homer needs to deliver Lisa's homework to her at school, and he must do so before Principal Skinner arrives. Suppose Homer starts from the Simpson home in his car and travels with velocity given by the figure below. Suppose that Principal Skinner passes the Simpson home on his bicycle 2 minutes after Homer has left, following him to the school. Principal Skinner is able to sail through all the traffic and travels with constant velocity 10 miles per hour.



(a) How far does Homer travel during the 10 minutes shown in the graph?.

To calculate how far Homer travels during the 10 minutes shown in the graph we find the area under the graph of Homer's velocity. Note that we must multiply by a constant so that the units are correct! This gives that distance $\frac{1}{60}$(area under curve)$= \frac{1}{60}(97.5) = 1.625$ miles.

(b) What is the average of Homer's velocity during the 10 minute drive?

average $= (\frac{1}{10}) \int_0^{10} v(t)dt = 9.75$ miles/hr.

(c) At what time, $t > 0$, is Homer the greatest distance ahead of Principal Skinner?

As long as Homer's velocity is greater than Principal Skinner's velocity, Homer is becoming farther away from Principal Skinner. Since Principal Skinner is traveling at a constant velocity of 10 miles/hr, Homer is the greatest distance ahead of Skinner at $t \approx 5.5$ minutes.

(d) Does Principal Skinner overtake Homer, and if so, when? Explain.

Principal Skinner will overtake Homer when the distance he has traveled is equal to the distance that Homer has traveled. Notice though that the area under Homer's velocity curve and the area under Principal Skinner's velocity curve overlap. So, they will have traveled the same distance when the area between Homer's velocity curve and Skinner's velocity curve from $t = 0$ to $t = 5.5$ equals the area between the two velocity curves from $t = 5.5$ to some time $t > 5.5$. Notice though that the area between the two curves from $t = 0$ to $= 5.5$ is greater than the area between the two curves from $t = 5.5$ to $t = 10$. So, Skinner does not overtake Homer. Or, more precisely, Principal Skinner travels $\frac{1}{60}(10)(8) = 1.33$ miles. This is less than the 1.625 miles that Homer traveled in the 10 minutes.

11. Parts (a) and (b) are independent of each other.

(a) Find an anti-derivative of

$$\frac{1+x^3}{\sqrt[3]{4+4x+x^4}}$$

Solution: Begin by noticing that the numerator, $1 + x^3$, is almost the derivative of $4 + 4x + x^4$.

So we are looking at an expression of the form $g^{-\frac{1}{3}}\ g'$.
Using our method of judicious guessing, our first guess is
$guess\ \#1 = (4 + 4x + x^4)^{\frac{2}{3}}$.

Now d/dx $guess\ \#1 = \frac{2}{3}(4 + 4x + x^4)^{\frac{2}{3}}(4 + 4x^3) = \frac{8}{3}(4 + 4x + x^4)^{\frac{2}{3}}(1 + x^3)$
Next our second, and final guess is:

$$guess\ \#2 = \mathbf{\frac{3}{8}(4 + 4x + x^4)^{\frac{2}{3}}}$$

(b) Suppose that y = h(x) is a twice differentiable function *defined for* $0 < x < 2\pi$ and that

$$\frac{d^2y}{dx^2} = \frac{(x-4)(x^2+1)\sin x}{3+\cos^8 x}$$

Find all points of inflection of y = h(x) and list intervals where *h* is *concave up* and *concave down on its domain.*

Solution: Performing a sign analysis on $\frac{d^2y}{dx^2}$ *, we see that the only transition points in the domain* $(0, 2\pi)$ *are* π *and 4.*

Now $\frac{d^2y}{dx^2} > 0$ *on* $(0, \pi)$ *and on* $(4, 2\pi)$*. Also* $\frac{d^2y}{dx^2} < 0$ *on* $(\pi, 4)$*.*

This means that h is concave up on the intervals $(0, \pi)$ *and* $(4, 2\pi)$ *and that h is concave down on* $(\pi, 4)$*. Consequently, h has* ***inflection points at*** $x = \pi$ ***and at*** $x = 4$***.***

12. *[University of Michigan]* Suppose that H(c) is the average temperature, in degrees Fahrenheit, that can be maintained in Oscar's apartment during the month of December as a function of the cost of the heating bill, c, in dollars. In complete sentences, give a practical interpretation of each of the following:

(a) H(50) = 65

Answer:

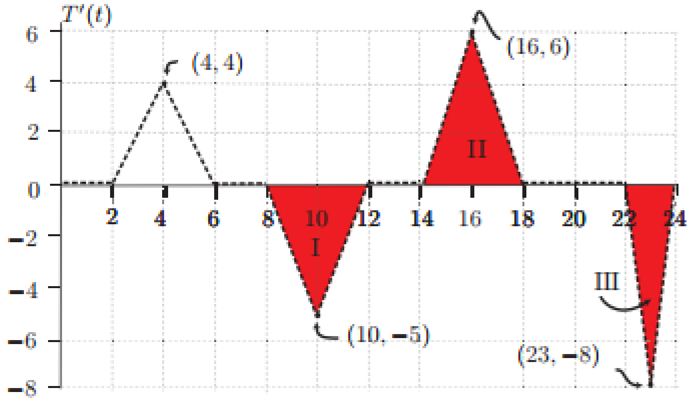
The practical interpretation of $H(50) = 65$ is that if Oscar's heating bill costs \$50.00 in December then he is able to maintain an average temperature during that month of 65°.

(b) $H'(50) = 2$

Answer:

The practical interpretation of $H'(50) = 2$ is that if Oscar's December heating bill increases from \$50.00 to \$51.00, then the average temperature he can maintain during that month will change from 65° to approximately 67°.

Suppose $T(t)$ gives the temperature in Oscar's apartment on December 18th in °F as a function of the time, t, in hours since 12:00 midnight. Below is a graph of $T'(t)$: **(NOTE: the graph is of $T'(t)$.)**



(c) When Oscar gets home from work at 6 pm the temperature in his apartment is 67 degrees. What was the temperature when he left for work at 8 am?

Solution:

Using the Fundamental Theorem of Calculus we know that

$$\int_8^{18} T'(t)dt = T(18) - T(8).$$

We are given that $T(18) = 67°$. Computing the areas I and II on the graph of T' we know that the $\int_8^{18} T'(t)dt$ =-II+I= $-10 + 12 = 2$ Consequently, $2 = 67° - T(8)$. Thus the temperature at 8 am is $T(8) = 65°$.

(d) If the temperature at 6 pm is 67 degrees, what is the minimum temperature in the apartment on December 18th?

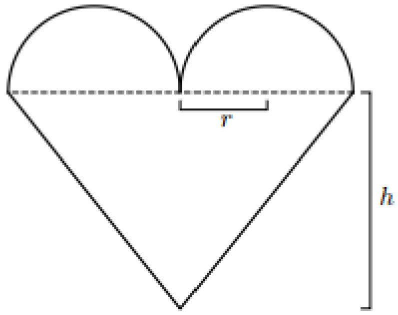
Solution:

When $T'(t) > 0$ ($T'(t) < 0$) then T is increasing (decreasing) and when $T'(x)$ is 0 on an interval the temperature is constant. Thus for $12 < t < 2$, $6 < t < 8$, $12 < t < 14$ and $18 < t < 22$ the temperature is not changing, for $2 < t < 6$ and $14 < t < 18$ the temperature is increasing, while it is decreasing for $8 < t < 12$ and $22 < t < 24$. Consequently the minimum temperature either occurs at $t = 0, t = 12$ or $t = 24$. Using the Fundamental Theorem of Calculus, and our knowledge of interpreting integrals as areas we have, $T(12) = T(18) - \int_{12}^{18} T'(t)dt = 67 - II = 67 - 12 = 55°$ and $T(24) = T(18) + \int_{18}^{24} T'(t)dt = 67 - III = 67 - 8 = 59°$, and similarly $T(0) = 59°$. Consequently, the minimum temperature occurs anywhere between noon and 2pm. This minimum temperature is 55°F.

EXTRA CREDIT:

[University of Michigan] For Valentine's Day, Marcel decides to make a heart-shaped cookie for Albertine to try to win her over. Being mathematically-minded, the only kind of heart that Marcel knows how to construct is composed of two half-circles of radius *r* and an isosceles triangle of height *h*, as shown below. Marcel happens to know that Albertine's love for him will be determined by the dimensions of the cookie she receives; if given a cookie as described above, her love L will be

$L = hr^2$, where r and h are measured in centimeters and L is measured in pitter-patters, a standard unit of affection. If Marcel wants to make a cookie whose area is exactly 300 cm^2, what should the dimensions be to maximize Albertine's love?



Solution: The area of the heart shape is

$$A = \pi r^2 + 2rh.$$

Setting this equal to 300 and solving for h gives the formula

$$h = \frac{300 - \pi r^2}{2r} = 150r^{-1} - \frac{\pi}{2}r.$$

Therefore, the formula for L can be written in terms of r alone:

$$L(r) = (150r^{-1} - \tfrac{\pi}{2}r)r^2 = 150r - \tfrac{\pi}{2}r^3.$$

We need to find the global maximum of $L(r)$. We have

$$L'(r) = 150 - \frac{3\pi}{2}r^2 = 0 \Rightarrow r = \frac{10}{\sqrt{\pi}}.$$

This critical point is a local maximum of L by the second-derivative test, since

$$L''(r) = -3\pi r \Rightarrow L''(10/\sqrt{\pi}) = -30\sqrt{\pi} < 0.$$

Since it is the only critical point, it must therefore be the global maximum.
Plugging this value of r into our formula, we can find the value of h, as well. We find that the dimensions that maximize Sophie's love are:

$$r = 10/\sqrt{\pi} \approx 5.6419\text{cm},$$

$$h = 10\sqrt{\pi} \approx 17.7245\text{cm}.$$