WORKSHEET XIII

HYPERBOLIC FUNCTIONS

The St. Louis arch is in the shape of a hyperbolic cosine.

Hyperbolic functions are very useful in both mathematics and physics. You may have already encountered them in precalculus. If not, here are their definitions:

$\sinh x = (e^x - e^{-x})/2$

$\cosh x = (e^x + e^{-x})/2$

$\tanh x = \sinh(x) / \cosh(x)$

$\coth x = 1/\tanh(x)$

$\text{sech } x = 1/\cosh(x)$

$\text{csch } x = 1/\sinh(x)$

Oddly enough, they enjoy certain similarities with the trigonometric functions, with which you are much more familiar.

1. Graph the six hyperbolic functions: sinh x, cosh x, tanh x, coth x, sech x, csch x. For each curve, determine the limit of *y* as *x* tends toward infinity or negative infinity. Which of the

functions are *odd*? which are *even*? (Remember that an *odd function* is one that is symmetric with respect to the origin; an *even function* is one that is symmetric with respect to the y-axis.)

2. Find the derivative of each of the six hyperbolic functions.

3. Expand cosh(x+y), cosh(2x), tanh(x+y), and tanh(2x).

4. Show that $(\cosh x)^2 - (\sinh x)^2 = 1$.

5. Show that $1 - (\tanh x)^2 = (\operatorname{sech} x)^2$.

6. Show that:

$$\cosh\frac{x}{2} = \sqrt{\frac{\cosh x + 1}{2}}$$

(*Note* that this corresponds to the half-angle formula for cosine. Similar formulas exist for sinh(x/2) and tanh(x/2).)

Hint: Compare the squares of each of the two sides.

7. Find the limit of $(\sinh x) / e^x$ as x tends toward infinity.

8. Simplify the expression:

$$\sinh\left(\ln\left(x + \sqrt{x^2 + 1}\right)\right)$$

Use your answer to find a formula for the inverse of sinh(x).

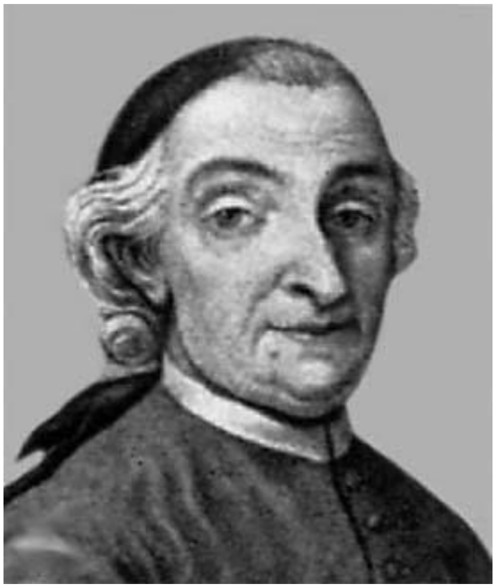
9. The inverse of sinh x in Mathematica is represented by ArcSinh[x]. Graph the curve y = ArcSinh(x). Find formulas for the derivative and the integral of arcsinh(x).

10. Repeat question 9 for the functions ArcCosh(x) and ArcTanh(x).

11. If the ends of a chain are attached to the points (-1, 0) and (1, 0) in the Cartesian plane, the chain will take the shape of the curve (called a *catenary*) given by:

$$f(x) = \frac{\cosh(a\,x) - \cosh a}{a}$$

where the constant *a* depends upon the length of the chain. Show that for any value of *a*, the graph of y = f(x) passes through the two points (-1, 0) and (1, 0).



Vincenzo Riccati (1707 - 1775) is given credit for introducing the hyperbolic functions.