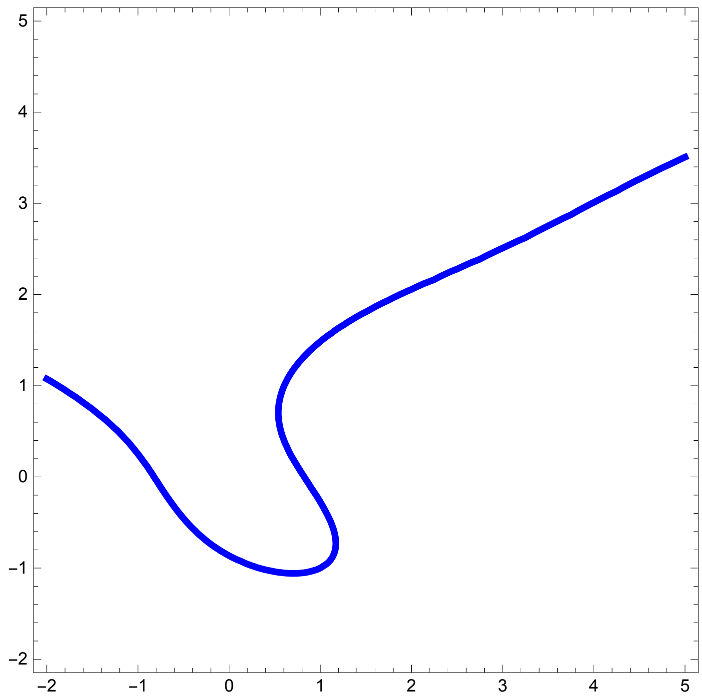
Implicit Differentiation

```
Clear[x, y, lhs, rhs]
```

Consider the implicitly defined curve $x^2 + xy - y^3 = 12 + \sin(x + y)$. Let us find the tangent line to this curve at the point $P = (1, -1)$

```
ContourPlot[x^2 + x y - y^3 == Cos[x + y], {x, -2, 5},
 {y, -2, 5}, ContourStyle → {Blue, Thickness[0.01]}]
```




```
lhs = D[x^2 + x y[x] - y[x]^3, x]
```

$2x + y[x] + x\,y'[x] - 3\,y[x]^2\,y'[x]$

```
rhs = D[Cos[x + y[x]], x]
```

$-\text{Sin}[x + y[x]]\,(1 + y'[x])$

```
Solve[lhs == rhs, y'[x]]
```

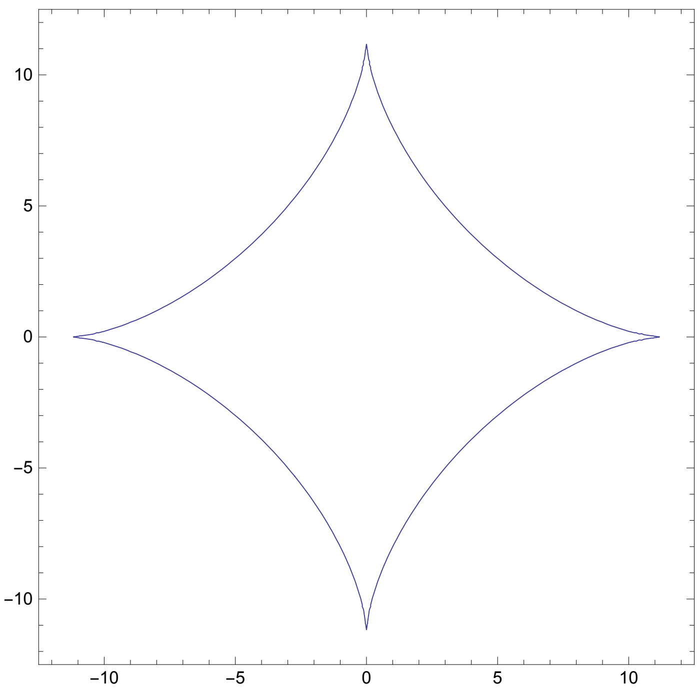
Thus, in standard notation, dy / dx is given by the expression :

$$-\frac{2x+\sin(x+y)+y}{x+\sin(x+y)-3y^2}$$

Next, let us find the tangent line to the Astroid, $x^{\frac{2}{3}}+y^{\frac{2}{3}} = 5$, at the point P = (8, 1)

```
Clear[x, y, lhs, rhs, m]
```

```
ContourPlot[ Abs[x]^(2/3) + Abs[y]^(2/3) == 5, {x, -12, 12}, {y, -12, 12}]
```




```
lhs = D[ x^(2/3) + y[x]^(2/3), x]
```

$$\frac{2}{3\,x^{1/3}}+\frac{2\,y'[x]}{3\,y[x]^{1/3}}$$

```
rhs = D[5, x]
```

0

```
Solve[lhs == rhs, y'[x]]
```

$$\left\{\left\{y'[x]\to -\frac{y[x]^{1/3}}{x^{1/3}}\right\}\right\}$$

To evaluate dy / dx at the point (8, 1) we define a function, h, of two variables

```
h[x_, y_] = - ∛y / ∛x
```

$-\frac{y^{1/3}}{x^{1/3}}$

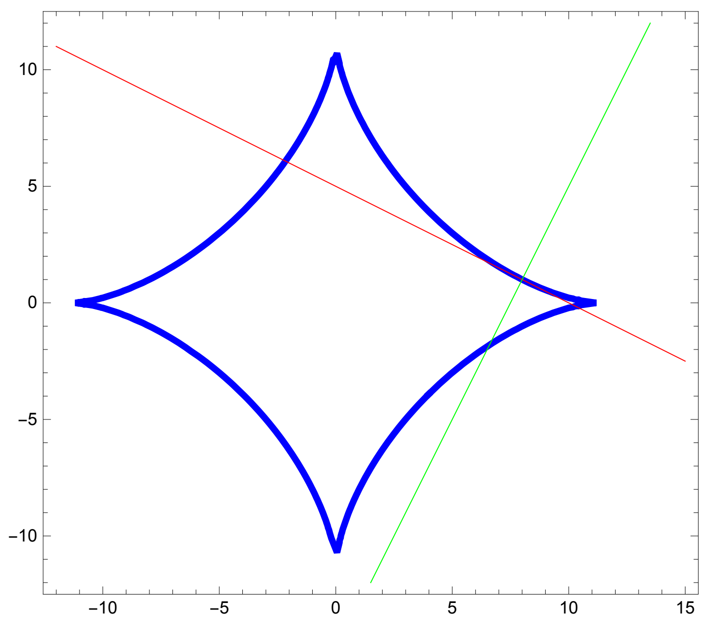
```
m = h[8, 1]
```

$-\frac{1}{2}$

So the equation of the tangent line to our implicitly defined curve at P = (8, 1) is given by : y – 1 = m (x – 8)

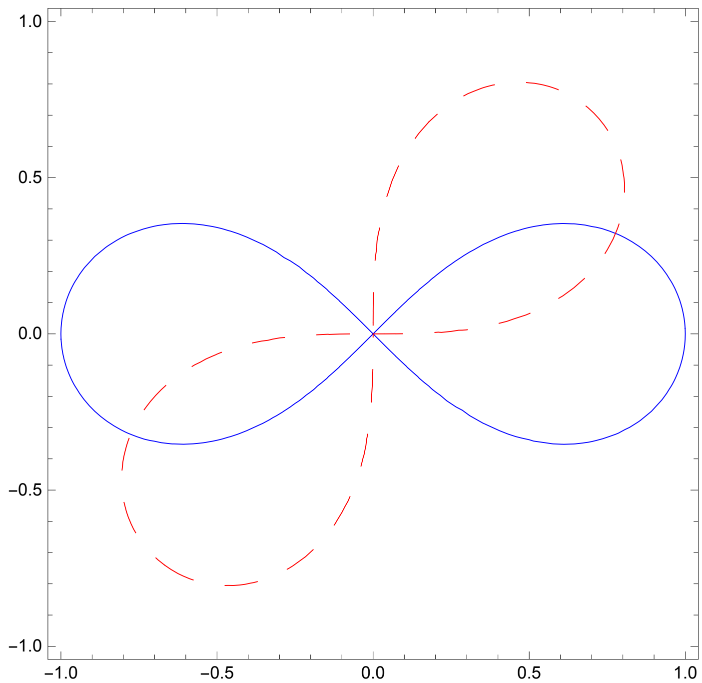
Now we will graph the Astroid together with its tangent and normal lines at the point P = (8, 1)

```
ContourPlot[ {Abs[x]^(2/3) + Abs[y]^(2/3) == 5, y - 1 == m (x - 8), y - 1 == (-1/m) (x - 8)},
 {x, -12, 15}, {y, -12, 12}, AspectRatio → Automatic,
 ContourStyle → {{Blue, Thickness[0.01]}, Red, Green}]
```

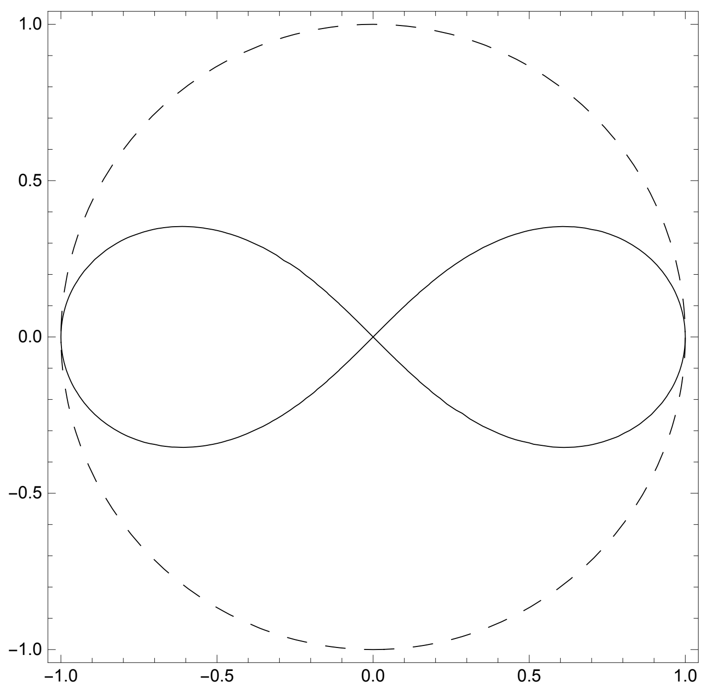


Below are more examples of plots of implicitly defined functions.

```
ContourPlot[{(x^2 + y^2)^2 == x^2 - y^2, (x^2 + y^2)^2 == 2 x y}, {x, -1, 1},
 {y, -1, 1}, ContourStyle → {Blue, {Red, Dashing[{.05}]}}]
```



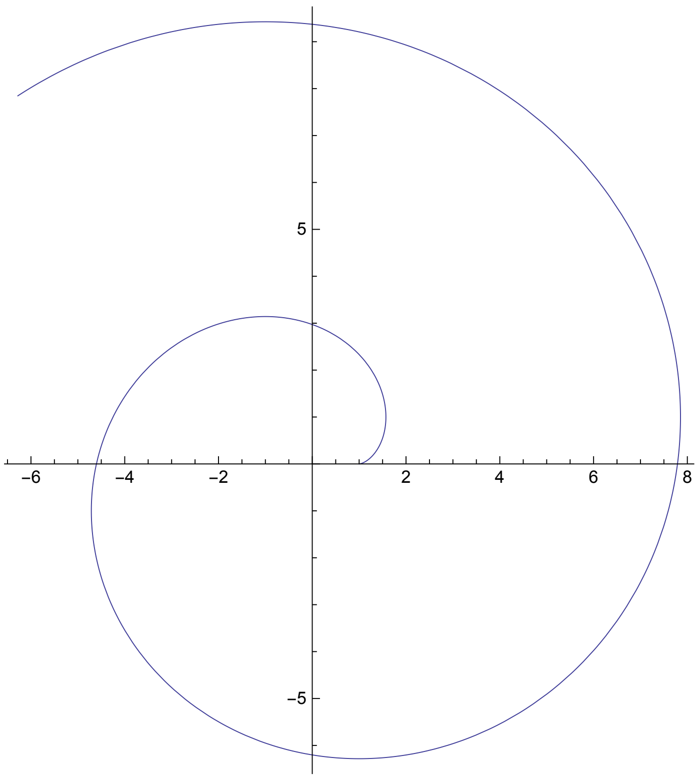

```
ContourPlot[{(x^2 + y^2)^2 == x^2 - y^2, x^2 + y^2 == 1}, {x, -1, 1},
 {y, -1, 1}, ContourStyle → {GrayLevel[0], Dashing[{.03}]}]
```



Parametric Equations

Consider the Involute of a circle, defined by x (t) = cos t + t sin t,
y (t) = sin t − t cos t, where 0 ≤ t ≤ 10.

```
ParametricPlot[{Cos[t] + t Sin[t], Sin[t] - t Cos[t]}, {t, 0, 10}]
```



To find the slope of the curve as a function of t,
we use the fact that dy / dx = (dy / dt) / (dx / dt), which follows from the Chain Rule.

```
Clear[f, g, slope]
```

```
f[t_] := Cos[t] + t Sin[t]

g[t_] := Sin[t] - t Cos[t]
```

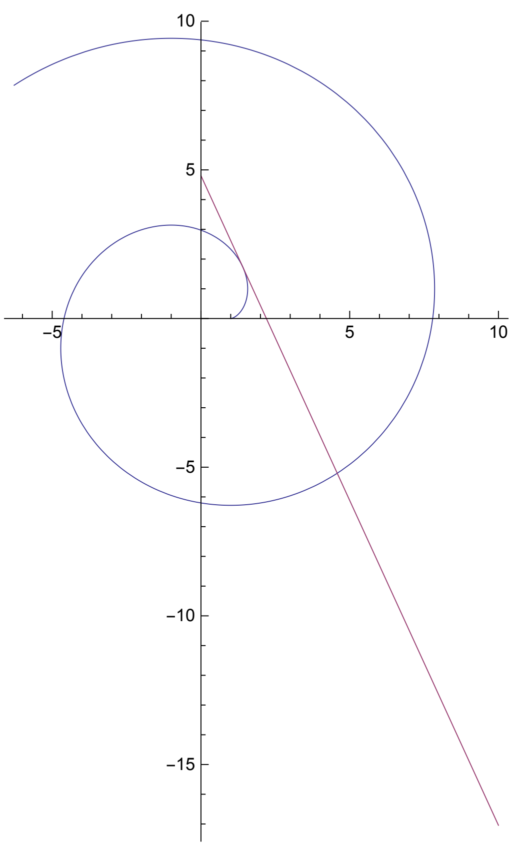
```
slope = D[g[t], t] / D[f[t], t]
```

```
Tan[t]
```

To plot the equation of the tangent line ($y = m x + b$) to the Involute of a circle at $t = 2$, we write the line in parametric form setting $x(t) = t$.

For the involute of a circle at $t = 2$: $x = t$, $y = \tan[2] (t - x[2]) + y(2)$

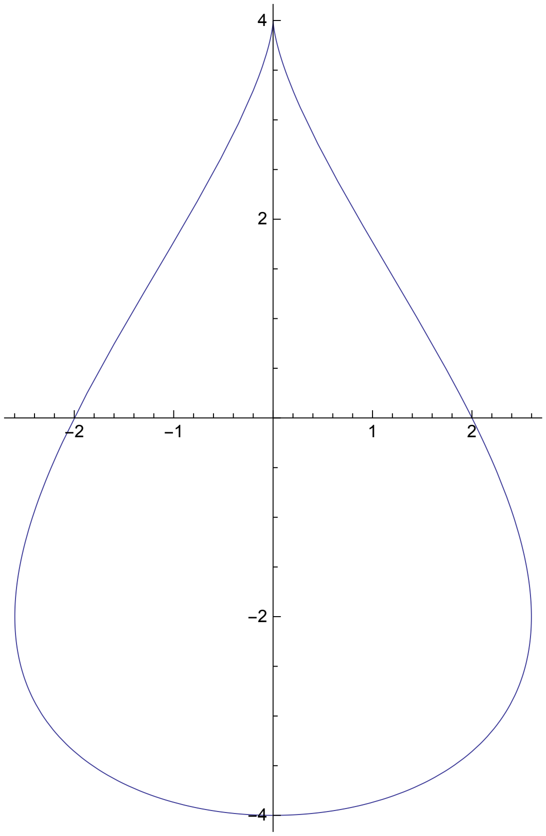
```
ParametricPlot[
 {{Cos[t] + t Sin[t], Sin[t] - t Cos[t]}, {t, Tan[2] (t - f[2]) + g[2]}}, {t, 0, 10}]
```



Next, consider the Teardrop, defined by $x(t) = 2 \cos t - \sin 2t$, $y(t) = 4 \sin t$

```
Clear[t, f, g, slope]
```

```
ParametricPlot[{2 Cos[t] - Sin[2 t], 4 Sin[t]}, {t, 0, 2 Pi}]
```



Here again, the slope of the curve at t is given by (dy / dt) / (dx / dt) :

```
f[t_] = 2 Cos[t] - Sin[2 t]
```

2 Cos[t] - Sin[2 t]

```
g[t_] = 4 Sin[t]
```

4 Sin[t]

$$\text{slope} = \frac{D[g[t], t]}{D[f[t], t]}$$

$$\frac{4\,\text{Cos}[t]}{-2\,\text{Cos}[2\,t] - 2\,\text{Sin}[t]}$$

Just for fun, here is a "neat example" of a parametric curve given in Mathematica :

```
ParametricPlot[{Sin[u] u , Cos[u] u},
 {u, 0, 100}, PlotPoints → 125, Axes → False,
 ColorFunction → (ColorData["Rainbow"][#3] &), MaxRecursion → 0, PlotStyle → Thick]
```