

MATH 161 CLASS DISCUSSION: 28TH NOVEMBER

HYPERBOLIC FUNCTIONS



The St. Louis arch is in the shape of a hyperbolic cosine.

Hyperbolic functions are very useful in both mathematics and physics. You may have already encountered them in pre-calculus. If not, here are their definitions:

$$\sinh x = (e^x - e^{-x})/2$$

$$\cosh x = (e^x + e^{-x})/2$$

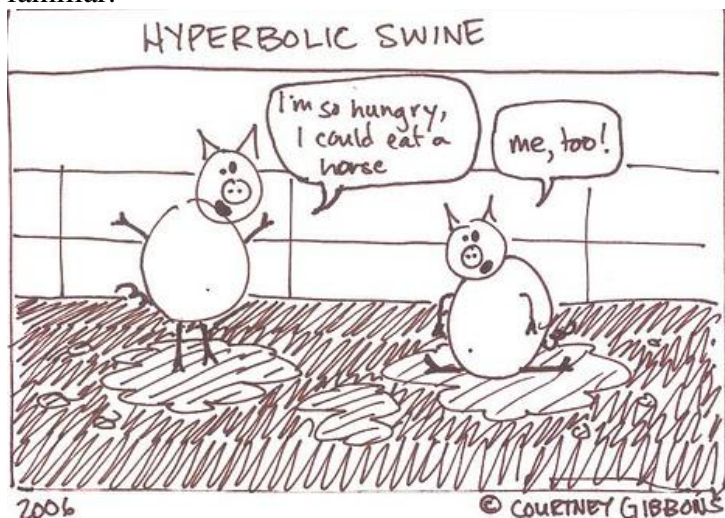
$$\tanh x = \sinh(x) / \cosh(x)$$

$$\coth x = 1/\tanh(x)$$

$$\operatorname{sech} x = 1/\cosh(x)$$

$$\operatorname{csch} x = 1/\sinh(x)$$

Oddly enough, they enjoy certain similarities with the trigonometric functions, with which you are much more familiar.



1. Graph the six hyperbolic functions: $\sinh x$, $\cosh x$, $\tanh x$, $\coth x$, $\operatorname{sech} x$, $\operatorname{csch} x$. For each curve, determine the limit of y as x tends toward infinity or negative infinity. Which of the functions are *odd*? which are *even*? (Remember that an *odd function* is one that is symmetric with respect to the origin; an *even function* is one that is symmetric with respect to the y -axis.)
2. Find the derivative of each of the six hyperbolic functions.
3. Expand $\cosh(x+y)$, $\cosh(2x)$, $\tanh(x+y)$, and $\tanh(2x)$.

4. Show that $(\cosh x)^2 - (\sinh x)^2 = 1$.
5. Show that $1 - (\tanh x)^2 = (\operatorname{sech} x)^2$.
6. Show that:

$$\cosh \frac{x}{2} = \sqrt{\frac{\cosh x + 1}{2}}$$

(Note that this corresponds to the half-angle formula for cosine. Similar formulas exist for $\sinh(x/2)$ and $\tanh(x/2)$.) *Hint:* Compare the squares of each of the two sides.

7. Find the limit of $(\sinh x) / e^x$ as x tends toward infinity.
8. Simplify the expression:

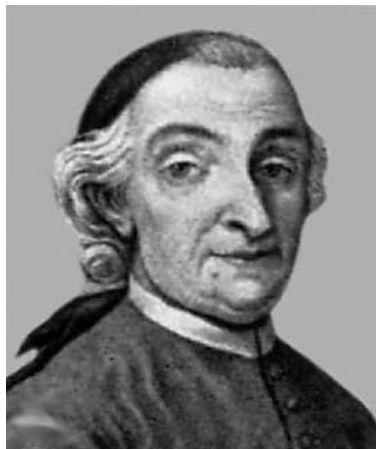
$$\sinh\left(\ln\left(x + \sqrt{x^2 + 1}\right)\right)$$

Use your answer to find a formula for the inverse of $\sinh(x)$.

9. The inverse of $\sinh x$ in Mathematica is represented by $\operatorname{ArcSinh}[x]$. Graph the curve $y = \operatorname{ArcSinh}(x)$. Find formulas for the derivative and the integral of $\operatorname{arcsinh}(x)$.
10. Repeat question 9 for the functions $\operatorname{ArcCosh}(x)$ and $\operatorname{ArcTanh}(x)$.
11. If the ends of a chain are attached to the points $(-1, 0)$ and $(1, 0)$ in the Cartesian plane, the chain will take the shape of the curve (called a *catenary*) given by:

$$f(x) = \frac{\cosh(ax) - \cosh a}{a}$$

where the constant a depends upon the length of the chain. Show that for any value of a , the graph of $y = f(x)$ passes through the two points $(-1, 0)$ and $(1, 0)$.



Vincenzo Riccati (1707 - 1775) is given credit for introducing the hyperbolic functions.