MATH 161 CLASS DISCUSSION 31 OCTOBER 2018

REVIEW

1. For each of the following functions, discuss concavity and locate any and all inflection points. Apply the second derivative test for local extrema. Sketch!

(a) $y = x^3 - 6x$

(b) $y = x^4 - 2x^3$

(c) $y = 2x - x^{1/3}$

(d) $y = x^2e^{-x}$

2. Applying the **Extreme Value Theorem**, find the global max and min values of each of the following functions defined on the given closed and bounded interval:

(a) $y = 3x^4 - 4x^3 - 8$ on $[-1, 2]$

(b) $y = 2 \cos x - x$ on $[0, 2\pi]$

(c) $y = -2x^3 + 3x^2 + 12x + 4$ on $[-2, 3]$

3. Given $f(x) = x^8 - 3x^5$ on the interval $[-1, 4]$.

(a) Find all critical points of *f*.

(b) Determine on which intervals *f* is increasing.

(c) Find and classify all local and global extrema of *f*.

(d) On which interval(s) is *f* concave up? Find all the points of inflection.

(e) Sketch the graph of *f* using the above information.

4. Sketch the curve $f(x) = (2x-1)^{13}(x-7)^{18}$ locating and classifying all critical points. Do not examine concavity.

5. Albertine is 2 miles offshore in a boat and wishes to reach a coastal village 6 miles down a straight shoreline from the point nearest the boat. She can row 2 mph and walk 5 mph. Where should she land her boat to reach the village in the least amount of time?

6. Find the point on the line x/a + y/b = 2 that is *closest* to the origin.

7. Swann is designing a rectangular poster to contain 50 in^2 of printing with a 4-inch margin at the top and bottom and a 2-inch margin at each side. Which overall dimensions minimize the amount of poster board used?

8. A rectangle is to be inscribed in a semicircle of radius 2 cm. What is the largest area the rectangle can have and what are its dimensions?

9. You are planning to make an open rectangular box from an 8-inch-by-15-inch piece of tin by cutting congruent squares from the corners and folding up the sides.
What are the dimensions of the box of largest volume you can construct this way and what is its volume?

10. (MIT practice test problem) A new junk food — NoKarb PopKorn — is to be sold in large cylindrical metal cans with a removable plastic lid instead of a metal top. The metal side and bottom will be of uniform thickness, and the volume is fixed to be 64π cubic inches. What base radius r and height h for the can will require the least amount of metal? Show work, and include an argument to show your values for r and h .

11. A plasma TV screen of height 36 inches is mounted on a wall so that its lower edge is twelve inches above eye-level of an observer. How far from the wall should the observer stand so that the viewing angle θ

subtended at her eye by the TV screen is as large as possible?

12. A grain silo has the shape of a right circular cylinder surmounted by a hemisphere. If the silo is to have a volume of 504 π ft^3, determine the radius and height of the silo that *requires the least amount of material* to build. 0

MIT 18.01 PRACTICE PROBLEMS

1. For the function $3x^5 - 5x^3 + 1$, sketch the graph over a suitable interval showing all the local maximum and minimum points on the graph, the points of inflection, and the approximate location of its zeros (show on which intervals of the form $[n, n+1]$, (n is an integer) they occur. Show work, or indicate reasoning.

2. Sketch the graph of $4x^2 - \dfrac{1}{x}$ over an interval showing its interesting features – local maxima and minima, points of inflection, zeros, asymptotes.

3. A line of negative slope through $(1, 2)$ cuts off a triangle in the first quadrant. For which such line will the triangle have least area? (Use its slope m as the independent variable. Show that you get a minimum.)

4. The bottom of the legs of a three-legged table are the vertices of an isosceles triangle with sides 5, 5, and 6. The legs are to be braced at the bottom by three wires in the shape of a Y. What is the minimum length of wire needed? Show it is a minimum.

PROBLEMS FROM STEWART:

2. Find two numbers whose difference is 100 and whose product is a minimum.

3. Find two positive numbers whose product is 100 and whose sum is a minimum.

4. The sum of two positive numbers is 16. What is the smallest possible value of the sum of their squares?

5. What is the maximum vertical distance between the line $y = x + 2$ and the parabola $y = x^2$ for $-1 \leqslant x \leqslant 2$?

6. What is the minimum vertical distance between the parabolas $y = x^2 + 1$ and $y = x - x^2$?

7. Find the dimensions of a rectangle with perimeter 100 m whose area is as large as possible.

8. Find the dimensions of a rectangle with area 1000 m^2 whose perimeter is as small as possible.

9. A model used for the yield Y of an agricultural crop as a function of the nitrogen level N in the soil (measured in appropriate units) is

$$Y = \frac{kN}{1 + N^2}$$

where k is a positive constant. What nitrogen level gives the best yield?

10. The rate (in mg carbon/m^3/h) at which photosynthesis takes place for a species of phytoplankton is modeled by the function

$$P = \frac{100I}{I^2 + I + 4}$$

where I is the light intensity (measured in thousands of foot-candles). For what light intensity is P a maximum?

11. Consider the following problem: A farmer with 750 ft of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible total area of the four pens?

(a) Draw several diagrams illustrating the situation, some with shallow, wide pens and some with deep, narrow pens. Find the total areas of these configurations. Does it appear that there is a maximum area? If so, estimate it.

(b) Draw a diagram illustrating the general situation. Introduce notation and label the diagram with your symbols.

(c) Write an expression for the total area.

(d) Use the given information to write an equation that relates the variables.

(e) Use part (d) to write the total area as a function of one variable.

(f) Finish solving the problem and compare the answer with your estimate in part (a).

12. Consider the following problem: A box with an open top is to be constructed from a square piece of cardboard, 3 ft wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box car have.

(a) Draw several diagrams to illustrate the situation, some short boxes with large bases and some tall boxes with small bases. Find the volumes of several such boxes. Does it appear that there is a maximum volume? If so, estimate it.

(b) Draw a diagram illustrating the general situation. Introduce notation and label the diagram with your symbols.

(c) Write an expression for the volume.

(d) Use the given information to write an equation that relates the variables.

(e) Use part (d) to write the volume as a function of one variable.

(f) Finish solving the problem and compare the answer wit your estimate in part (a).

13. A farmer wants to fence in an area of 1.5 million square feet in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. How can he do this so as to minimize the cost of the fence?

14. A box with a square base and open top must have a volume 32,000 cm^3. Find the dimensions of the box that minimize tl amount of material used

15. If 1200 cm^2 of material is available to make a box with a square base and an open top, find the largest possible volum of the box.

16. A rectangular storage container with an open top is to have a volume of 10 m^3. The length of its base is twice the width. Material for the base costs \$10 per square meter. Material fc the sides costs \$6 per square meter. Find the cost of material for the cheapest such container.

17. Do Exercise 16 assuming the container has a lid that is mad from the same material as the sides.

18. A farmer wants to fence in a rectangular plot of land adjace to the north wall of his barn. No fencing is needed along the barn, and the fencing along the west side of the plot is share with a neighbor who will split the cost of that portion of the fence. If the fencing costs \$20 per linear foot to install and the farmer is not willing to spend more than \$5000, find the dimensions for the plot that would enclose the most area.

19. If the farmer in Exercise 18 wants to enclose 8000 square feet of land, what dimensions will minimize the cost of the fence?

20. (a) Show that of all the rectangles with a given area, the on with smallest perimeter is a square.

(b) Show that of all the rectangles with a given perimeter, tl one with greatest area is a square.

21. Find the point on the line $y = 2x + 3$ that is closest to the origin.

22. Find the point on the curve $y = \sqrt{x}$ that is closest to the point (3, 0).

23. Find the points on the ellipse $4x^2 + y^2 = 4$ that are farthest away from the point $(1, 0)$.

24. Find, correct to two decimal places, the coordinates of the point on the curve $y = \sin x$ that is closest to the point $(4, 2)$.

25. Find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius r.

26. Find the area of the largest rectangle that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$.

27. Find the dimensions of the rectangle of largest area that can be inscribed in an equilateral triangle of side L if one side of the rectangle lies on the base of the triangle.

28. Find the area of the largest trapezoid that can be inscribed in a circle of radius 1 and whose base is a diameter of the circle.

29. Find the dimensions of the isosceles triangle of largest area that can be inscribed in a circle of radius r.

30. If the two equal sides of an isosceles triangle have length a, find the length of the third side that maximizes the area of the triangle.

31. A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible volume of such a cylinder.

32. A right circular cylinder is inscribed in a cone with height h and base radius r. Find the largest possible volume of such a cylinder.

33. A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible surface area of such a cylinder.

34. A Norman window has the shape of a rectangle surmounted by a semicircle. (Thus the diameter of the semicircle is equal to the width of the rectangle. See Exercise 1.1.62.) If the perimeter of the window is 30 ft, find the dimensions of the window so that the greatest possible amount of light is admitted.

35. The top and bottom margins of a poster are each 6 cm and the side margins are each 4 cm. If the area of printed material on the poster is fixed at 384 cm^2, find the dimensions of the poster with the smallest area.

36. A poster is to have an area of 180 in^2 with 1-inch margins at the bottom and sides and a 2-inch margin at the top. What dimensions will give the largest printed area?

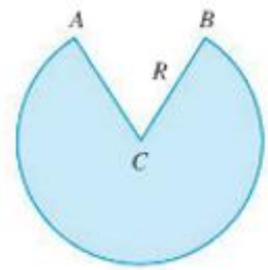
37. A piece of wire 10 m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is (a) a maximum? (b) A minimum?

38. Answer Exercise 37 if one piece is bent into a square and the other into a circle.

39. If you are offered one slice from a round pizza (in other words, a sector of a circle) and the slice must have a perimeter of 32 inches, what diameter pizza will reward you with the largest slice?

40. A fence 8 ft tall runs parallel to a tall building at a distance of 4 ft from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?

41. A cone-shaped drinking cup is made from a circular piece of paper of radius R by cutting out a sector and joining the edges CA and CB. Find the maximum capacity of such a cup.



42. A cone-shaped paper drinking cup is to be made to hold 27 cm^3 of water. Find the height and radius of the cup that will use the smallest amount of paper.

43. A cone with height h is inscribed in a larger cone with height H so that its vertex is at the center of the base of the larger cone. Show that the inner cone has maximum volume when $h = \frac{1}{3}H$.

44. An object with weight W is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ with a plane, then the magnitude of the force is

$$F = \frac{\mu W}{\mu \sin\theta + \cos\theta}$$

where μ is a constant called the coefficient of friction. For what value of θ is F smallest?

45. If a resistor of R ohms is connected across a battery of E volts with internal resistance r ohms, then the power (in watts) in the external resistor is

$$P = \frac{E^2R}{(R + r)^2}$$

If E and r are fixed but R varies, what is the maximum value of the power?

46. For a fish swimming at a speed v relative to the water, the energy expenditure per unit time is proportional to v^3. It is believed that migrating fish try to minimize the total energy required to swim a fixed distance. If the fish are swimming against a current u $(u < v)$, then the time

UNIVERSITY OF MICHIGAN

1. Saruman, the new U. S. Secretary of the Interior, is creating an army of orcs to cut down all the trees in Fangorn Forest. Saruman is currently trying to decide exactly how large the army should be to destroy the forest as quickly as possible. The trouble is that orcs are not very efficient. In very small armies they tend to work pretty well — one orc will emerge as the leader, and he will have good control over the others. They also organize fairly well in very large armies, once a military structure is established. In medium-sized armies, though, the orcs spend a lot of time fighting for dominance, and as a result, they cannot work very efficiently. Saruman has noticed this, of course. His research indicates that an army of x thousand orcs, will be able to cut down.

$$T(x) = \frac{x^3}{3} - 3x^2 + 8x \qquad \text{thousand trees per hour.}$$

(a) If Saruman is capable of producing an army of up to 3000 orcs, how many should he produce to maximize the hourly destruction of trees? (Saruman does not have a graphing calculator and must be convinced by the methods of calculus.)

(b) Does your answer change if Saruman can produce up to 4000 orcs? If so, how many should he produce now?

(c) Does your answer change if Saruman can produce up to 6000 orcs? If so, how many should he produce now?

2. Elphaba has found a corrupt prison guard, Mert, to sell her metal piping to use to dig a tunnel out of prison. Mert can sell Elphaba steel piping and copper piping, and he provides the following information.

- The number of kilograms (kg) of soil that Elphaba can dig with steel piping is proportional to the number of centimeters (cm) of steel piping that she buys. She can dig 50 kg of soil per cm of steel piping, and her cost (in dollars) of buying x cm of steel piping is given by $A(x) = x^2 + x$.

- The number of kilograms (kg) of soil that Elphaba can dig with copper piping is proportional to the number of centimeters (cm) of copper piping that she buys. She can dig 30 kg of soil per cm of copper piping, and her cost (in dollars) of buying y cm of copper piping is given by $B(y) = 2y$.

(a) How many kilograms of soil can Elphaba dig with *x* cm of steel piping?

For parts (b) and (c) below, suppose Elphaba buys *w* cm of steel piping and *k* cm of copper piping and that this is exactly the right amount of piping so that she can dig through 2700 kg of soil to dig her escape tunnel.

(b) Write a formula for *k* in terms of *w*.

(c) Let T(w) be the total cost (in dollars) of all the piping Elphaba buys to dig her escape tunnel. Find a formula for the function T(w). The variable *k* and the function names A and B should not appear in your answer. (Note that T(w) is the function one would use to minimize Elphaba's costs. You should *not* do the optimization in this case.)

(d) What is the domain of T(w) in the context of this problem?