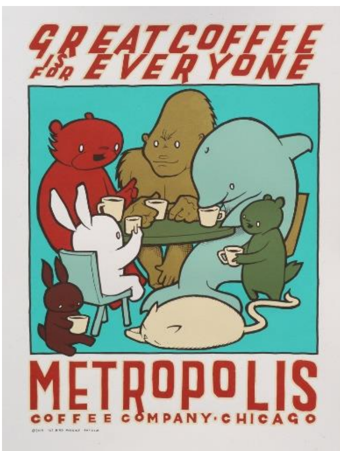
MATH 161 SOLUTIONS: TEST I

SEPTEMBER 2017

Instructions: *Answer any 10 of the following 12 questions. You may solve more than 10 to obtain extra credit.* ☺

1. Albertine orders a large cup of coffee at Metropolis on Granville. Let F(t) be the temperature in *degrees Fahrenheit* of her coffee *t minutes* after the coffee is placed on her tray.

(a) *(2 pts)* Explain the meaning of the statement: F(9) = 167. (Use complete sentences. Avoid any mathematical terms!)



Solution: Nine minutes after the cup of coffee is placed upon the tray, Albertine finds that the temperature of the coffee is 167°F.

(b) *(3 pts)* Explain the meaning of the statement: $F^{-1}(99) = 17.5$

Solution: When the temperature of the coffee is 99°F, 17.5 minutes have elapsed since the coffee mug was placed upon Albertine's tray.

(c) *(3 pts)* Give the *practical* interpretation of the statement: $F'(9) = -1.10$. (Use complete sentences. Do not use the words "derivative" or "rate" or any other mathematical term in your explanation.)

Solution: Nine minutes after being served her coffee, Albertine notices that the temperature of the coffee is decreasing by about 1.1°F every minute during the next couple of minutes.

(d) *(1 pt)* What are the *units* of $F'(9)$?

Answer: °F/ minute (since $\Delta F/\Delta t$ represents change in temp/ change in time.)

(e) *(3 pts)* Using the information given in parts (a) and (c), estimate the temperature of Albertine's coffee *seven* minutes after she has been handed the coffee.

*Solution: Since after nine minutes the temperature is 167°F, we would estimate that two minutes earlier the temperature was 167 + 2(1.1) = **169.2 °F**.*

(f) *(3 pts)* EXTRA CREDIT

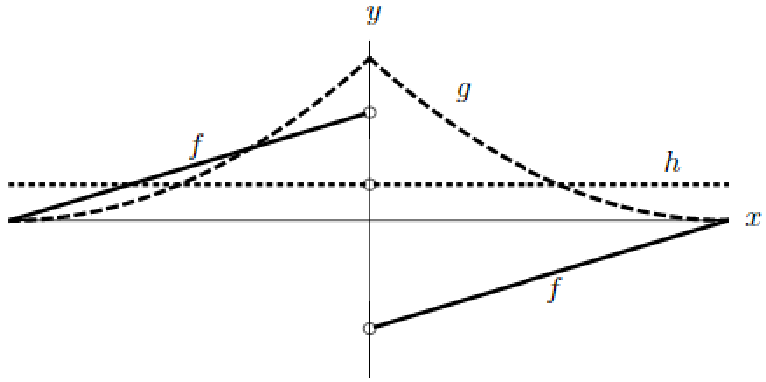
Explain the meaning of the statement:

$$(F^{-1})'(99) = -1$$

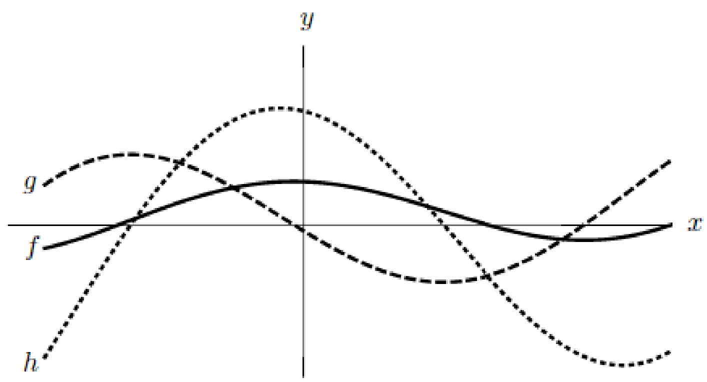
Solution: When the temperature of the coffee is 99 ºF, time increases by one minute for each ºF that the temperature of the coffee falls.

2. *(4 pts each)* For each of the following three sets of axes, exactly one of the following statements (a) – (e) is true. You may use a letter more than once. In the space provided next to each figure, enter the letter of the true statement for that figure. For each graph, note that the x and y scales are not the same.

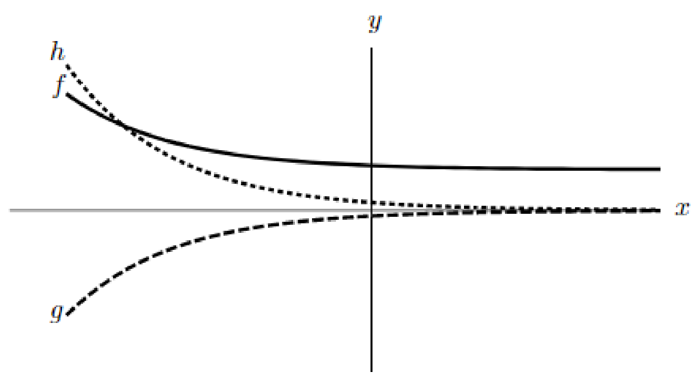
(a) h is the derivative of f, and f is the derivative of g.
(b) g is the derivative of f, and f is the derivative of h.
(c) g is the derivative of h, and h is the derivative of f.
(d) h is the derivative of g, and g is the derivative of f.
(e) None of **(a)**-**(d)** are possible.



True Statement: A



True Statement: E



True Statement: D

3. *(12 pts)* Using the limit definition of the derivative, write an explicit expression for the *derivative* of the function

$$g(x) = (\cos x)^x \text{ at } x = 3.$$

Do not try to calculate this derivative.

Solution:

$$g'(3) = \lim_{h\to 0} \frac{g(h+3) - g(3)}{h} = \lim_{h\to 0} \frac{(\cos(h+3))^{h+3} - (\cos 3)^3}{h}$$

4. *(a) (6 pts)* *Find* $\lim_{x\to\infty} f(x)$ *if, for all* $x > 5$,

$$\frac{4x-1}{x} < f(x) < \frac{4x^2+3x}{x^2}$$

Explain!

Which theorem are you using?

Solution: Since $\frac{4x-1}{x} \to 4$ *as* $x \to \infty$ *and* $\frac{4x^2+3x}{x^2} \to 4$ *as* $x \to \infty$*, we can invoke the **Squeeze Theorem** to conclude that f(x)* $\to 4$ *as* $x \to \infty$.

(b) *(6 pts)* Show that $y = f(x) = x^3 + 5e^x + 1$ has *at least one* real root. *Explain!*

Which theorem are you using?

Solution: Note that y = f(x) is continuous on the real line since it is a sum of continuous functions.

Next, notice that f(1) > 0 and f(-10) = -8.99 < 0.

*Invoking the **Intermediate Value Theorem**, we conclude that f must have a root on the interval (-10, 1).*

5. *(12 pts)* Find an equation of the *normal line* to the curve

$$y = g(x) = \frac{x^2-1}{x^2+1} \text{ at } x = 1.$$

You may use short cuts.

Solution: Let y = g(x). Then $g'(x) = \frac{(x^2+1)2x-(x^2-1)2x}{(x^2+1)^2}$.

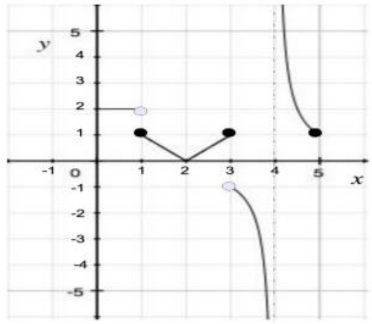
Thus $g'(1) = \frac{(1^2+1)2-(1^2-1)2}{(1^2+1)^2} = \frac{4}{4} = 1$.

Also note that g(0) = 0, so the point of tangency is (1, 0).

Finally, the slope of the normal line is -1 and the equation of the normal line is y – 0 = -(x – 1); more simply,

***y = 1 – x**.*

6. *(12 pts)* The graph of a function g is shown below. For each of the following, decide if the limit exists. If it does, find the limit. If it does not, decide also if the "limit" is ∞, -∞, or neither. No justification is necessary for full credit, but show your work for purposes of partial credit.



(a) $\lim_{x\to 0^+} g(x) = 2$

(b) $\lim_{x\to 1^-} g(x)$=*2*

(c) $\lim_{x\to 1^+} g(x)$=*1*

(d) $\lim_{x\to 1} g(x)$=*does not exist; neither* ± ∞.

(e) $\lim_{x\to 2} g(x)$= *0*

(f) $\lim_{x\to 0^+} g(x) =$ *[redundant]*

(g) $\lim_{x\to 3^-} g(x)$=*1*

(h) $\lim_{x\to 3^+} g(x) = -1$

(i) $\lim_{x\to 3} g(x) =$ *does not exist; neither* ± ∞.

(j) $\lim_{x\to 4^+} g(x) = \infty$

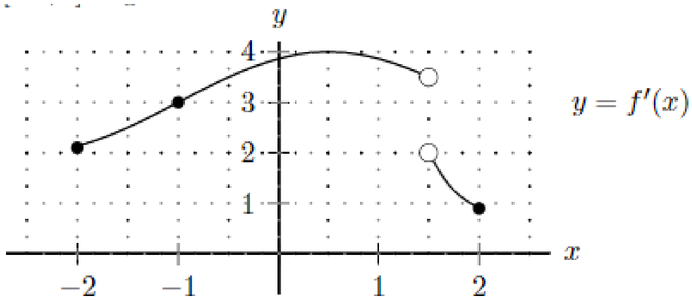
(k) $\lim_{x\to 4^-} g(x) =$ -∞

(l) $\lim_{x\to 4} g(x) =$ *does not exist; neither* ± ∞.

(m) $\lim_{x\to 5^-} g(x) =$ *1*

7. Suppose that f(x) is a function that is continuous on the interval [-2, 2]. The graph of

$f'(x)$ *on the interval* $[-2, 2]$ *is given below.*



(a) *(6 pts)* Let y = L(x) be the local linearization of f(x) at x = -1. Using the fact that f(-1) = -4, write a formula for y = L(x).

Solution: Note that f(-1) = - 4 and $f'(-1) = 3$. *So* $L(x) - (-4) = 3(x - (-1))$.

Simplifying: L(x) = -4+3(x+1)

(b) *(6 pts)* Use your formula for L(x) to approximate f(-0.5).

Solution: Since 0.5 is close to x = -1, our estimate is:

f(-0.5) ≈ L(-0.5) = -4 + 3(-0.5+1) = ***-5.5***

8. Suppose that *f* and *g* are differentiable functions satisfying:

f(3) = -2, g(3) = -4, f ′(3) = 3, and g′(3) = -1.

(a) *(6 pts)* Let H(x) = (f(x) + 2g(x) + 1)(f(x) – g(x) – 4). Compute H′(3) (Hint: Use short cuts here.)

Solution: Using the product rule,

$$H'(x) = \big(f'(x) + 2g'(x)\big)\big(f(x) - g(x) - 4\big) + \big(f(x) + 2g(x) + 1\big)\big(f'(x) - g'(x)\big)$$

So $H'(3) = \big(f'(3) + 2g'(3)\big)\big(f(3) - g(3) - 4\big) + \big(f(3)\big) + 2g(3) + 1\big)\big(f'(3) - g'(3)\big) =$

(3+(-2)(-2-(-4)-4) + (-2 + (-8) +1)(3-(-1)) = 1(-2) + (-9)(4) = ***-38***

(b) (6 pts) Let $M(x) = \dfrac{2f(x) + 3g(x)}{2 - 3g(x)}$. Compute M′(3)

Solution: Using the quotient rule,

$$M'(x) = \frac{(2-3g(x))\big(2f'(x)+3g'(x)\big)-(2f(x)+3g(x))(-3g'(x))}{(2-3g(x))^2}$$

$$\text{So, } M'(3) = \frac{(2-3g(3))\big(2f'(3)+3g'(3)\big)-(2f(3)+3g(3))(-3g'(3))}{(2-3g(3))^2} = \frac{14(3)-(-16)(3)}{14^2} = \frac{90}{196} = \mathbf{\frac{45}{98}}$$

9. For each of the following, find any and all critical points. Then, using the first derivative test, classify them (local max, local min, neither).

(a) *(6 pts)* $y = f(x) = x^3 - 3x + 1$

Solution: $\frac{dy}{dx} = 3x^2 - 3 = 3(x+1)(x-1)$

Hence the critical points are x = ±1.

Next, performing a sign analysis on dy/dx, we find that dy/dx is positive when |x| > 1 and negative when |x|<1. Hence f is rising on (-∞, -1), falling on (-1, 1), and rising on (1, ∞).

This means that f achieves a ***local maximum*** *at x = -1 and a* ***local minimum*** *at x =1.*

(b) *(6 pts)* $y = g(x) = 3x^4 - 16x^3 + 18x^2 + 1$

Solution: $\frac{dy}{dx} = 12x^3 - 48x^2 + 36x = 12x(x^2 - 4x + 3) = 12x(x-3)(x-1)$

Hence the critical points are x = 0, 1, 3.

Next, performing a sign analysis on dy/dx, we find that dy/dx is positive when x>3 and 0<x<1; dy/dx is negative when 1<x<3 and when x<0. . Hence g is rising on (0, 1) and on (3, ∞); g is falling on (-∞, 0) and on (1, 3).

This means that g achieves a ***local maximum*** *at x = 4 and* ***local minima*** *at x = 1 and x = 6.*

10. Let y = f(x) be a differentiable function with derivative

$$f'(x) = \frac{e^x(x-1)(x-2)^2(x-4)^3(x-5)^4(x-6)^5}{1+x^4}$$

(a) *(4 pts)* Find any and all critical points.

Solution: $f'(x) = 0$ *implies that* $x = 1, 2, 4, 5,$ *and* $6.$ These are the five critical points.

(b) *(8 pts)* Classify each critical point (local max, local min, neither).

Solution: Clearly since their respective factors have even exponents, x = 2 and x =5 are neither max nor min. Doing a sign analysis on $f'(x)$ *we find that the only transition points are* $x = 1, 4,$ *and* $6.$

Using the first derivative test, we find that x = 4 is a local maximum; x= 1 and x = 6 are both local minima.

11. *(3 pts each)* Compute each of the following limits. *Justify your reasoning.*

(*a*) $\lim_{x\to\infty} \frac{(4x^3+11)^2(3x-91)^3}{(2x^2+5)^4(2x+2017)}$

Solution: Observe that:

$$\frac{(4x^3+11)^2(3x-91)^3}{\left(2x^2+5\right)^4(2x+2017)} \cong \frac{(4x^3)^2(3x)^3}{(2x^2)^4(2x)} = \frac{16(27)}{32}\left(\frac{x^9}{x^9}\right) \to \frac{27}{2} \ as\ x \to \infty$$

$(b)\quad \lim_{x\to 3} \dfrac{\frac{1}{x^2}-\frac{1}{9}}{x-3}$

Solution: Observe that, as long as x ≠ 3:

$$\frac{\frac{1}{x^2}-\frac{1}{9}}{x-3} = \frac{\frac{9}{9x^2}-\frac{x^2}{9x^2}}{x-3} = \frac{9-x^2}{9x^2(x-3)} =$$

$$\frac{-(x-3)(3+x)}{9x^2(x-3)} = \frac{-(3+x)}{9x^2} \to -\frac{6}{81} = -\frac{2}{27} \ as\ x \to 2$$

$(c)\quad \lim_{x\to\infty} \dfrac{\sin 7x}{x}$

Solution: This limit is 0 due to the squeeze theorem, viz,

$$-1 \le \sin 7x \le 1$$

Since we can assume x > 0,

$$-\frac{1}{x} \le \frac{\sin 7x}{x} \le \frac{1}{x}$$

Now 1/x and -1/x →0 as x→ ∞, and so must $\frac{\sin 7x}{x}$.

$(d)\quad \lim_{x\to 0} \dfrac{\sqrt{x+1}-1}{x}$

Solution: We begin by rationalizing the numerator of the algebraic expression. Then we assume that, as long as x ≠ 0:

$$\frac{\sqrt{x+1}-1}{x} = \left(\frac{\sqrt{x+1}-1}{x}\right)\left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+2}\right) = \frac{x}{(x)(\sqrt{x+1}+2)} =$$

$$\frac{1}{\sqrt{x+1}+1} \to \frac{1}{\sqrt{1}+1} = \frac{1}{2} \ \ as\ x \to 0.$$

12. *(3 pts each)* For each of the following functions, determine the type of discontinuity at the given point. If it is a *removable* discontinuity, find a continuous extension of the function.

(a) $y = \frac{x^3-x^2-2x}{(x-2)(x+5)}$ at x = 2

Solution: Factoring yields

$$y = \frac{x(x^2-x-2)}{(x-2)(x+5)} = \frac{x(x-2)(x+1)}{(x-2)(x+5)} = \frac{x(x+1)}{x+5} \ provided\ that\ x \neq 2.$$

Now as x →2, y → $\frac{6}{7}$. *Thus this is a* ***removable discontinuity****. The continuous extension requires that y be defined as* $\frac{6}{7}$ *when x = 2.*

(b) $y = \frac{x^3-x^2-2x}{(x-2)(x+5)}$ at x = -5

Solution: In part (a), we factored y as follows:

$$y = \frac{x(x^2-x-2)}{(x-2)(x+5)} = \frac{x(x-2)(x+1)}{(x-2)(x+5)} = \frac{x(x+1)}{x+5} \ provided\ that\ x \neq 2.$$

Now as x → -5, the numerator tends to 20 but the denominator → 0. Thus this is an ***infinite discontinuity****.*

(c) $y = cos\frac{3}{x}\ \ at\ x = 0$

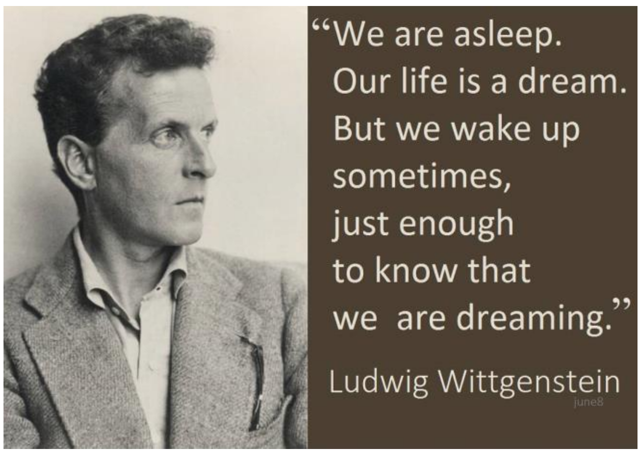
Solution: This is our archetypal example of an ***essential discontinuity****.*

(d) $y = \frac{|x|}{x}\ \ at\ x = 0$

Solution: This is a ***jump discontinuity*** *since* $\lim_{x \to 0^+} \frac{|x|}{x} = 1\ \ and\ \ \lim_{x \to 0^-} \frac{|x|}{x} = -1.$

It is a hypothesis that the sun will rise tomorrow: and this means that we do not know whether it will rise.

- Ludwig Wittgenstein



DERIVATIVE RULES

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

$$\frac{d}{dx}\left(a^x\right) = \ln a \cdot a^x$$

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = f(x)\cdot g'(x) + g(x)\cdot f'(x)$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{\left(g(x)\right)^2}$$

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x))\cdot g'(x)$$

$$\frac{d}{dx}\left(\sin x\right) = \cos x$$

$$\frac{d}{dx}\left(\cos x\right) = -\sin x$$

$$\frac{d}{dx}\left(\tan x\right) = \sec^2 x$$

$$\frac{d}{dx}\left(\cot x\right) = -\csc^2 x$$

$$\frac{d}{dx}\left(\sec x\right) = \sec x \tan x$$

$$\frac{d}{dx}\left(\csc x\right) = -\csc x \cot x$$

$$\frac{d}{dx}\left(\arcsin x\right) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\arctan x\right) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\operatorname{arc\,sec} x\right) = \frac{1}{x\sqrt{x^2-1}}$$