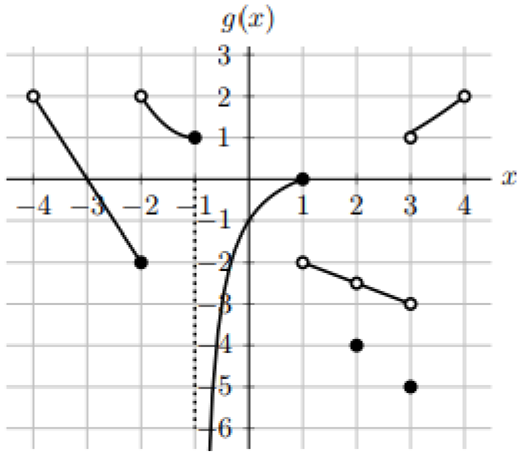
MATH 161 SOLUTIONS: QUIZ III

1. Consider the functions f(x) and g(x) given by the formula and graph below.

$$f(x) = \begin{cases} 2x^3 - 2x^2 & \text{for } x \le 1, \\ x^3 + 1 & \text{for } x > 1. \end{cases}$$



(a) Circle the correct answer(s) in each of the following questions.

i) At which of the following values of x is the function g(x) *not* continuous?

$x = -3$ $\boxed{x = -1}$ $x = 0$ $\boxed{x = 2}$ $x = 3.5$

ii) At which of the following values of x is the function f(x) + g(x) continuous?

$x = -2$ $x = -1$ $\boxed{x = 0}$ $\boxed{x = 1}$ $x = 2$

Note that g(x) is linear on each of the intervals (−4, −2), (1, 2) and (2, 3). All your answers below should be *exact*. If any of the quantities do not exist, write DNE.

(b) *Find* $\lim_{x \to 2} (2f(x) + g(x))$

Solution: $\lim_{x \to 2} (2f(x) + g(x)) = 2 \lim_{x \to 2} f(x) + \lim_{x \to 2} g(x) =$

$$2(9) + (-2.5) = 15.5$$

(c) *Find* $\lim_{x \to \infty} \frac{f(2x)}{x^3}$

Answer: **8**

(d) *Find* $\lim_{x \to \infty} g(x^2 e^{-x} + 3)$

Answer: **1**

(e) *For which value(s) of p does* $\lim_{x \to p+} g(x) = 1$

Answer: **-3.5, 3**

(f) *Find* $\lim_{x \to -1^-} f(-x)$

Solution: Let t = -x. Then as $x \to -1^-, t \to 1^+$.

$Thus \lim_{x\to -1^-} f(-x) = \lim_{t\to 1^+} f(t) = \mathbf{2}$

2. Using the IVT explain why the function f(x) = x + 3 – 2 sin x must have *at least one real root.*

Solution: *Note that f(0) = 3 > 0 and $f(-\pi)$= - π + 3 – 2 sin $(-\pi)$ = 3 – π < 0.*

Also note that f is continuous on the interval [$-\pi$, 0] since f is a sum of continuous functions. Since f($-\pi$) < 0 < f(0), the IVT guarantees the existence of a root of the equation G(x) = 0 in the interval [$-\pi$, 0].

3. Using the *Squeeze Theorem*, show that the function

$$f(x) = 3x^4 \cos\left(\frac{x+1732}{x^3}\right)$$

has a limit as x ⟶ 0 and find the value of this limit. (You need not state the general theorem; only show how it can be applied here.)

Solution: *First, a brief review of the Squeeze theorem.*

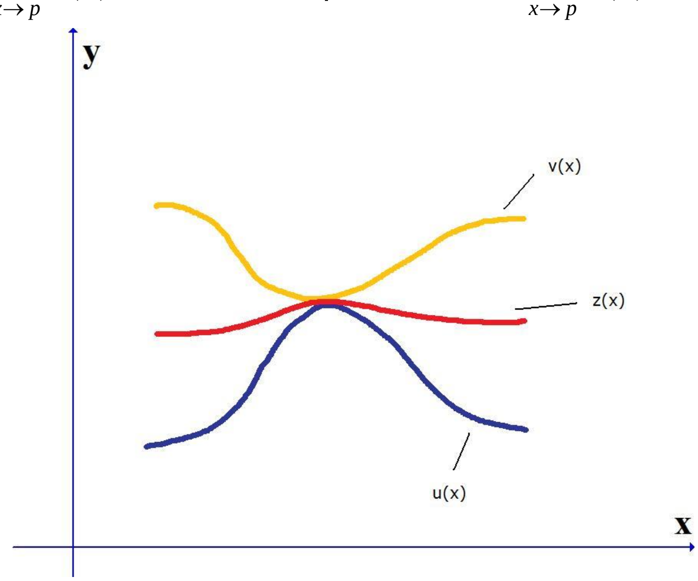
Theorem: Let v(x), z(x), u(x) be functions defined on an interval (a, b) except possibly at the point p, where p ∈(a, b)

In addition, assume that

$$\lim_{x\to p} v(x) = L \text{ exists} \quad and \quad \lim_{x\to p} u(x) = L \text{ exists}.$$

Then

$$\lim_{x\to p} z(x) = exists \quad and, \; furthermore, \; \lim_{x\to p} z(x) = L.$$



Returning to the original question:

$$-1 \le \cos\left(\frac{x}{x^3+1732}\right) \le 1 \Rightarrow$$

$$-3x^4 \le 3x^4 \cos\left(\frac{x}{x^3+1732}\right) \le 3x^4$$

Now, as $x \to 0$, $3x^4 \to 0$ and $-3x^4 \to 0$.

Thus, invoking the Squeeze Theorem, we obtain:

$$\lim_{x\to 0} 3x^4 \cos\frac{x+1732}{x^3} = 0$$

4. Compute each of the following limits. As usual, show your work

(a) Find $\lim_{x\to\frac{\pi}{2}} \frac{\sin x}{x}$

Solution: *Since* $\lim_{x\to\pi/2} x \neq 0$ we can use the limit law for quotients:

$$\lim_{x\to\frac{\pi}{2}} \frac{\sin x}{x} = \frac{\lim_{x\to\pi/2} \sin x}{\lim_{x\to\pi/2} x} = \frac{1}{\left(\frac{\pi}{2}\right)} = \frac{2}{\pi}$$

(b) Find $\lim_{x\to 0} \sqrt{\frac{\sin 9x}{x}}$

Solution:

$$\lim_{x\to 0} \sqrt{\frac{\sin 9x}{x}} = \sqrt{\lim_{x\to 0} \frac{\sin 9x}{x}} = \sqrt{9} = 3$$

(c) Find $\lim_{x\to 0} \frac{\tan(21x)}{\text{x}\cos(3x)}$

Solution:

$$\lim_{x\to 0} \frac{\tan(21x)}{\text{x}\cos(3x)} = \lim_{x\to 0} \frac{\sin(21x)}{\text{x}\cos(3x)\cos 21x} =$$

$$\lim_{x\to 0} \frac{1}{\cos(3x)\cos 21x} \lim_{x\to 0} \frac{\sin(21x)}{x} =$$

$$(1)(21) = 21$$

(d) Find $\lim_{x\to\infty} \frac{\sin(7x+3)}{7x+3}$

Solution: *Since* $-1 \le \sin(7x+3) \le 1$ *it follows that*

$$\frac{-1}{7x+3} \le \frac{\sin(7x+3)}{7x+3} \le \frac{1}{7x+3} \quad \textit{for all } x>0.$$

Noting that $\lim_{x\to\infty} \frac{1}{7x+3} = 0$, *we may apply the Squeeze Theorem to conclude:*

$$\lim_{x\to\infty} \frac{\sin(7x+3)}{7x+3} = 0$$

5. Consider the rational function *F* defined by $F(x) = \dfrac{2x^3 + x^2 - 3x}{2x^2 - 5x - 12}$

(a) Where is *F* undefined? (*Hint:* Your answer should consist of two x values.)

Solution: Begin by factoring: $F(x) = \frac{x(2x+3)(x-1)}{(2x+3)(x-4)}$

Now F is undefined when its denominator is 0:

This occurs at x = -3/2 and at x = 4.

(b) Let *p* denote the smaller of the two numbers found in part (a). Is it possible to extend *F* to a function that is continuous at x = p? If not, explain; if so, how should F(p) be defined?

Solution: Clearly p = -3/2.

Since $F(x) = \frac{x(2x+3)(x-1)}{(2x+3)(x-4)}$, *it follows that, when* $x \neq -\frac{3}{2}$,

$$F(x) = \frac{x(x-1)}{x-4} \to \frac{-\frac{3}{2}(-\frac{3}{2}-1)}{-\frac{3}{2}-4} = -\frac{15}{22} \text{ as } x \to -\frac{3}{2}.$$

So, since this limit exists, the function ***has a continuous extension at x = -3/2.***

I*n other words, x = -3/2 is a* ***removable discontinuity****.*

Furthermore, we should assign the value ***-15/22*** *to F(-3/2).*

(c) Let *q* denote the larger of the two numbers found in part (a). Is it possible to extend *F* to a function that is continuous at x = q? If not, explain; if so, how should F(q) be defined?

Solution: Clearly q = 4.

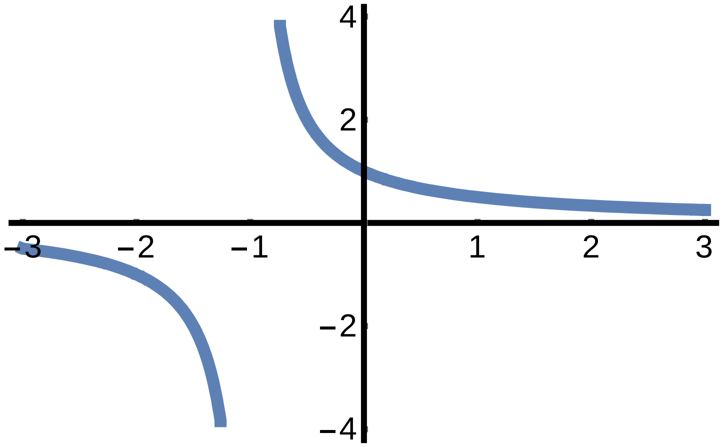
Now as $x \to 4$,

the numerator of F approaches 12 *while the denminator of F approaches* 0.

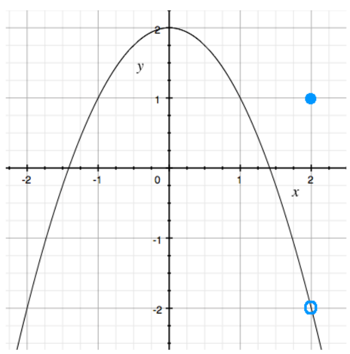
Thus we have an infinite discontinuity, implying there is no continuous extension of F at x = 4.

6. Give the *type* of discontinuity for each function below at the given point.

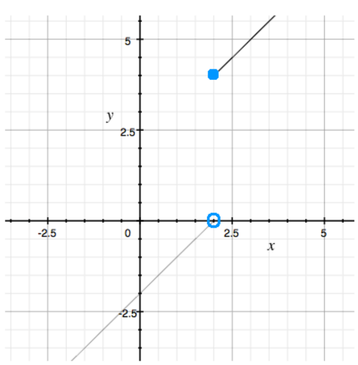
(a) At x = -1 answer: infinite discontinuity



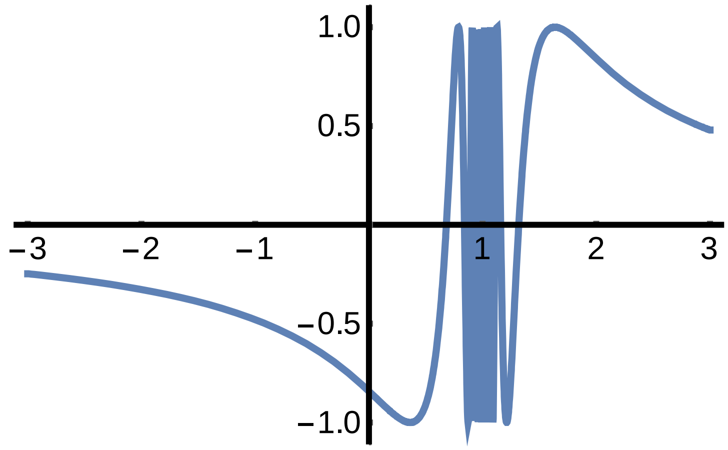
(b) At x = 2 answer: removable discontinuity



(c) At x = 2 answer: jump discontinuity



(d) At x = 1 answer: essential discontinuity



It is a hypothesis that the sun will rise tomorrow: and this means that we do not know whether it will rise.

- Ludwig Wittgenstein