MATH 161 SOLUTIONS: QUIZ VIII 2 NOVEMBER 2018

1. *[14 pts]* Let $g(x) = 3x^5 – 5x^3$ be defined on the real line.

(a) Find all the *critical points* of *g*.

Solution: Since $g'(x) = 15x^4 - 15x^2 = 15x^2(x+1)(x-1)$, the critical points of g are x = 0, 1, -1

(b) Where is *g rising*? (Give the appropriate intervals.)

Solution: Performing a sign analysis on $g'(x)$ we discover that g rises on $(-\infty, -1)$ *and on* $(1, \infty)$.

(c) Find and classify all local extrema. Justify your answer. (Do not say "because the calculator tells me.")

Solution: Using the first-derivative test, we discover that *g* has a local maximum at x = -1 and a local minimum at x = 1.

(d) Does g achieve a global max or global min? Explain.

Solution: Neither a global max nor a global min. In the long run, g(x) behaves as x^5 which is unbounded above as well as unbounded below.

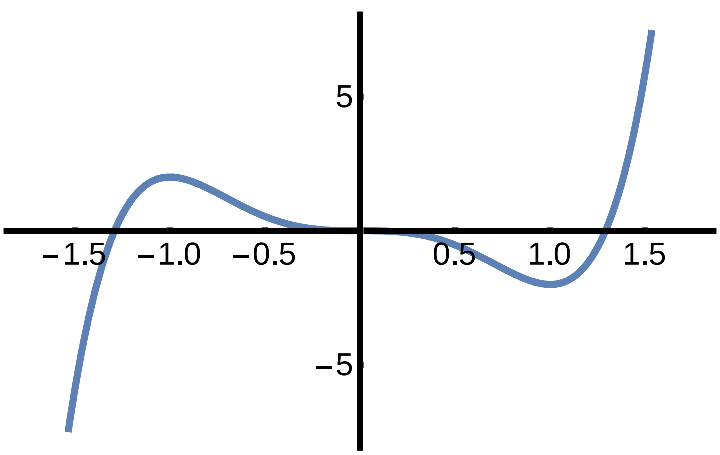
(e) Where is *g concave up*? (Give the appropriate intervals.) Find any and all *points of inflection*.

Solution: $g''(x) = \frac{d}{dx}(15x^4 - 15x^2) = 60x^3 - 30x = 30x(2x^2 - 1) = 60x\left(x + \sqrt{\frac{1}{2}}\right)\left(x - \sqrt{\frac{1}{2}}\right)$.

Performing a sign analysis on g'', we find three points of inflection: $x = 0, x = \pm\frac{\sqrt{2}}{2}$

Moreover, g is concave up on $\left(-\frac{\sqrt{2}}{2}, 0\right)$ *and* $\left(\frac{\sqrt{2}}{2}, \infty\right)$.

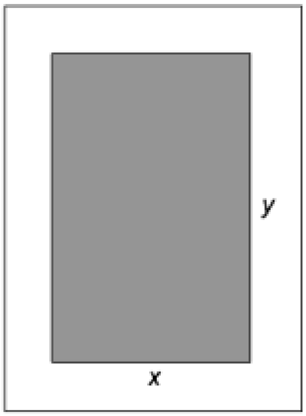
(f) Sketch a graph of y = g(x). Label **all local and global extrema** and **all inflection points**. Show regions of increase and decrease. Show regions where the function is concave up and concave down.



2. *[8 pts]* Albertine is designing a rectangular poster to contain 50 in^2 of printing with a 4-inch margin at the top and bottom and a 2-inch margin at each side. Which overall dimensions *minimize* the amount of poster board used?

Solution: Let x = width of the printed region (in inches) and let y = length of printed region (in inches).

Hence the outer width is x + 4 inches and the outer length is y + 8 inches.



We are asked to minimize A = (x + 4)(y + 8).

We area given that xy = 50, from which we see that $y = \frac{50}{x}$.

Thus $A = (x+4)\left(\frac{50}{x} + 8\right) = 82 + 8x + \frac{200}{x}$.

The appropriate domain of A is $(0, \infty)$.

Next

$$\frac{dA}{dx} = 8 - \frac{200}{x^2} = 8\left(1 - \frac{25}{x^2}\right) = \frac{8(x+5)(x-5)}{x^2}$$

So the critical points are $x = \pm 5$. Of course, we reject -5, so the unique critical point in our domain occurs at x = 5.

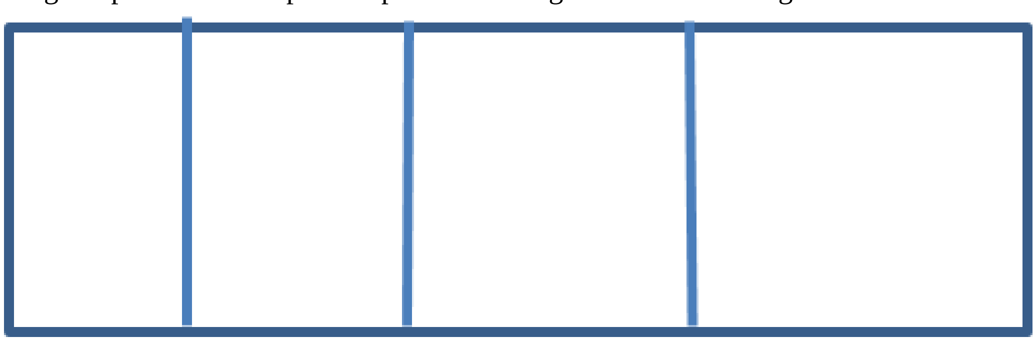
And $here\ y = \frac{50}{x} = 10$.

The first derivative test verifies that this is a local minimum, and hence, in our domain, a global minimum.

Hence the overall dimensions are:

9 inches by 18 inches.

3. *[8 pts]* Build a rectangular pen with three parallel partitions using 500 feet of fencing. What dimensions will *maximize* the



total area of the pen?

Solution: Let x = length (in ft) of the pen and let y = width (in ft) .

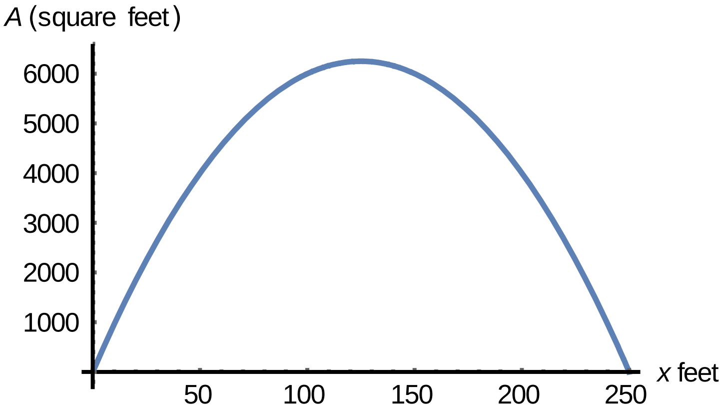
Then the total amount of fencing is 500 = 2x + 5y.

We wish to maximize the area of the pen: A = xy.

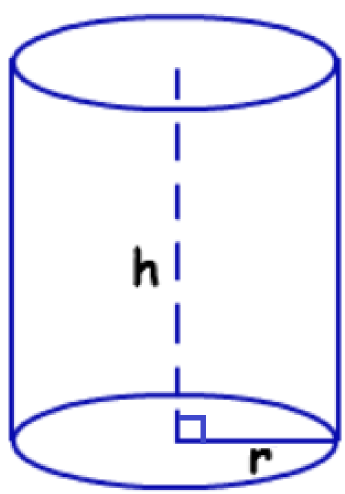
Now $y = \frac{500-2x}{5}$. The constraints on x are: $0 < x < 250$.

So $A = xy = \frac{x(500-2x)}{5}$. Next, $\frac{dA}{dx} = \frac{1}{5}(500-4x) = \frac{4}{5}(125-x)$.

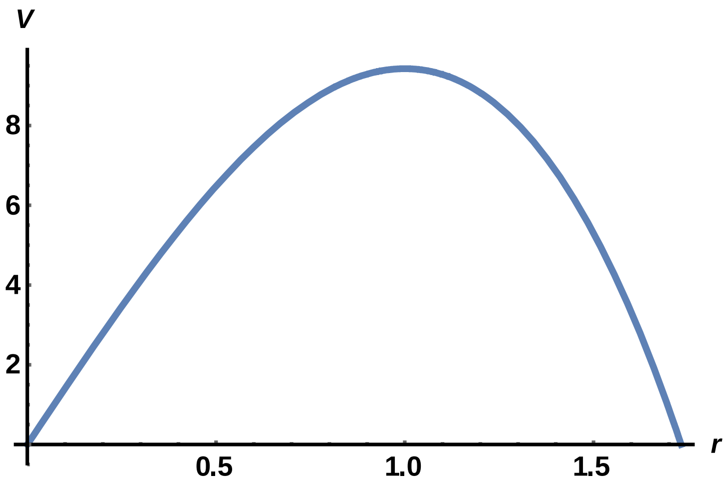
This concave-down parabola has a maximum at **x = 125 feet. And at x = 125, y = 50 feet.**



4. *[8 pts]* A container in the shape of a right circular cylinder with no top has surface area 3π ft^2. What height h and base radius r will maximize the volume of the cylinder?



Solution: We are given that $3\pi = 2\pi rh + \pi r^2$, or equivalently, $3 = 2rh + r^2$, from which $h = \frac{3-r^2}{2r}$.
The constraints on r are $0 < r < \sqrt{3}$
We are asked to maximize $V(r) = \pi r^2 h = \pi r^2 \left(\frac{3-r^2}{2r}\right) = \frac{\pi}{2} r(3-r^2)$ subject to the above constraints.
Now $\frac{dV}{dr} = \frac{\pi}{2}(3-3r^2) = \frac{3\pi}{2}(1-r)(1+r)$. Now the unique critical point in our domain is r = 1.
We can easily confirm, using the first derivative test, that r = 1 is a global max in our domain. Now, when r = 1, h = 1.
So the volume of the cylinder is maximized when **r = 1 ft and h = 1 ft.**



EXTRA CREDIT *[University of Michigan problem]*

The cable of a suspension bridge with two supports $2L$ meters apart hangs H meters above the ground. The height H is given in terms of the distance in meters from the first support x (in meters) by the function

$$H(x) = e^{x-L} + e^{L-x} + H_0 - 2$$

where H_0 and L are positive constants. Notice that x ranges from 0 (the first support) to $2L$ (the second support).

(a) Find (but do not classify) the critical points for the function *H(x).*

Solution: To find the critical points, we first take the derivative of $H(x)$:

$$H'(x) = e^{x-L} - e^{L-x}.$$

To find the critical points, we set $H'(x) = 0$:

$$H'(x) = 0 \quad \Rightarrow \quad e^{x-L} - e^{L-x} = 0 \Rightarrow e^{x-L} = e^{L-x}$$

apply ln to both sides $\quad x - L = L - x$

$$2x = 2L \Rightarrow x = L$$

So, $H(x)$ has only one critical point at $x = L$.

(b) Find the *x* and *y* coordinates of all global maxima and minima for the function *H(x)*. Justify your answers!

Solution: Since the values of x lie in the closed interval $[0, 2L]$, to find all global maxima and minima, we need to compare the values of H at the endpoints and at any critical points. From part (a), we know the only critical point is at $x = L$, we plug $x = 0, L$, and $2L$ into $H(x)$:

$H(0) = e^{-L} + e^{L} + H_0 - 2$, $H(L) = 1 + 1 + H_0 - 2 = H_0$, and $H(2L) = e^{L} + e^{-L} + H_0 - 2$.

To identify which of these should be larger, we notice that $e^{x} + e^{-x} \geq 2$ for all x. Therefore, $H(2L) = H(0) > H(L)$. Then, the function $H(x)$ has global maxima at $(0, e^{-L} + e^{L} + H_0 - 2)$ and $(2L, e^{-L} + e^{L} + H_0 - 2)$ and a global minimum at (L, H_0).