MATH 162 SOLUTIONS: QUIZ III

1. Explain why the following improper integral diverges:

$$\int_e^{\infty} \frac{1}{\sqrt{\ln x}}\,dx$$

Solution:

First note that $x > \ln x$ *for all* $x \geq e$. *Hence:*

$$\sqrt{x} > \sqrt{\ln x}$$

and so:

$$0 < \frac{1}{\sqrt{x}} \leq \frac{1}{\sqrt{\ln x}} \quad \textit{for } x \geq e$$

Recalling that

$$\int_e^{\infty} \frac{1}{x^{1/2}}\,dx$$

diverges by the p-test, we now invoke the Comparison Test to obtain the desired result.

2. Compute the value of the following convergent improper integral. Assume that b is a positive constant.

$$\int_0^{\infty} e^{-bx}dx$$

Solution:

Using the definition of improper integral, we find:

$$\int_0^\infty e^{-bx}dx = \lim_{n\to\infty}\int_0^n e^{-bx}dx = \lim_{n\to\infty}\left(-\frac{1}{b}e^{-bx}\right)\Bigg|_0^n =$$

$$\lim_{n\to\infty}\left(-\frac{1}{b}\right)\left(e^{-bn} - e^{-0}\right) = \lim_{n\to\infty}\frac{1}{b}\left(1-\frac{1}{e^{bn}}\right) = \frac{1}{b}$$

3. Evaluate the following convergent improper integral. Show your work! Calculator solutions are not acceptable.

$$\int_0^\infty \frac{x}{\left(x^2+3\right)^{3/2}}\,dx$$

Using the definition of improper integral:

$$\int_0^\infty \frac{x}{\left(x^2+3\right)^{3/2}}\,dx = \lim_{n\to\infty}\int_0^n \frac{x}{\left(x^2+3\right)^{3/2}}\,dx = \lim_{n\to\infty}\int_0^n x\left(x^2+3\right)^{-\frac{3}{2}}\,dx =$$

$$\lim_{n\to\infty}\left(-\left(x^2+3\right)^{-\frac{1}{2}}\right)\Bigg|_0^n = \lim_{n\to\infty}\left(-\left(n^2+3\right)^{-1/2} + 3^{-1/2}\right) = \frac{1}{\sqrt{3}}$$

4. Evaluate the following convergent improper integral. Show your work! Calculator solutions are not acceptable.

$$\int_0^\infty \frac{\arctan x}{1+x^2}dx$$

Solution:

Using the definition of improper integral:

$$\int_0^\infty \frac{\arctan x}{1+x^2}dx = \lim_{n\to\infty} \frac{1}{2}(\arctan x)^2\Big|_0^n =$$

$$\frac{1}{2}\lim_{n\to\infty}\left((\arctan n)^2 - 0\right) = \frac{1}{2}\left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{8}$$

For each of the following improper integrals, determine convergence or divergence. *Justify each answer! (That is, if you use the comparison test, exhibit the function that you choose to use for comparison and show why the appropriate inequality holds.)* Calculator solutions are not acceptable.

5. $$\int_{13}^\infty \frac{13+x+x^2}{(2013+x)^4}\,dx$$

To apply the comparison test, observe that, for all $x \geq 13$:

$$0 \le \frac{13 + x + x^2}{(2013 + x)^4} \le \frac{13x^2 + x^2 + x^2}{x^4} = \frac{15x^2}{x^4} = 15\frac{1}{x^2}$$

Applying the p-test, the improper integral

$$\int_{13}^{\infty} \frac{1}{x^2}\, dx$$

converges, and hence, invoking the Comparison Test, the original improper integral must converge.

6. $$\int_{13}^{\infty} \frac{13 + x + e^x}{2013 + x^5 + 13e^x}\, dx$$

Solution:

To apply the comparison test, observe that, for all $x \ge 13$:

$$\frac{13 + x + e^x}{2013 + x^5 + 13e^x} \ge \frac{e^x}{2013e^x + e^x + e^x} = \frac{1}{2015} > 0$$

Since, the improper integral

$$\int_{13}^{\infty} \frac{1}{2015}\, dx$$

clearly diverges, the original improper integral must diverge as well.

Extra Extra Credit:

$$\int_0^{1-} \frac{1}{\sqrt{1-x^4}}\ dx$$ *(Hint: Try using the Comparison Test.)*

Solution:

Since $x^2 > x^4$ for $0 \le x < 1$, $1 - x^2 < 1 - x^4$, and thus $0 < \frac{1}{\sqrt{1-x^4}} < \frac{1}{\sqrt{1-x^2}}$ *for $0 \le x < 1$.*

Now $\int_0^{1-} \frac{1}{\sqrt{1-x^2}} dx = \lim_{b \to 1-} \int_0^{b} \frac{1}{\sqrt{1-x^2}} dx = \lim_{b \to 1-} (\arcsin b - \arcsin 0) = \frac{\pi}{2}$.

Thus, invoking the Comparison Test, the original integral converges also.