SOLUTIONS: QUIZ IV

1. Select *any three* of the following four integrals. For each improper integral that you select, determine convergence or divergence. *Justify your answers!* (You may answer all four for extra credit.)

(a) $$\int_0^{3-} \frac{1}{(3-x)^4}\,dx$$

This integral diverges since:

$$\int_0^{3-} \frac{1}{(3-x)^4}\,dx = \lim_{c\to 3-} \int_0^{c} \frac{1}{(3-x)^4}\,dx = \lim_{c\to 3-} \left. \frac{1}{3(3-x)^3} \right|_0^c =$$

$$\lim_{c\to 3-} \left(\frac{1}{3(3-c)^3} - \frac{1}{3(3-0)^3} \right) = \infty$$

(b) $$\int_{0+}^{1} \frac{1+5x+7x^{15}}{\sqrt{x}}\,dx$$

Note that the dominant term in the numerator is 1 (not x^{15}).

This integral converges because:

$$0 < \frac{1+5x+7x^{15}}{\sqrt{x}} \le \frac{13}{\sqrt{x}} \quad \textit{for } 0 < x \le 1$$

Using the p-test, we know that $\int_{0+}^{1} \frac{1}{\sqrt{x}}\,dx$ *converges. Hence, by the comparison test,*

$$\int_{0+}^{1} \frac{1+5x+7x^{15}}{\sqrt{x}}\,dx \quad \textit{converges}.$$

$$(c) \quad \int_{0}^{\frac{\pi}{4}-} \tan 2x \, dx$$

This integral diverges because:

$$\int_{0}^{\frac{\pi}{4}-} \tan 2x \, dx = \lim_{c\to\frac{\pi}{4}-} \int_{0}^{c} \tan 2x \, dx = \lim_{c\to\frac{\pi}{4}-} \frac{1}{2}(-\ln|\cos 2x|)\Big|_{0}^{c} =$$

$$-\frac{1}{2}\lim_{c\to\frac{\pi}{4}-}\left(\ln|\cos 2c| - \ln(\cos 0)\right) = -\frac{1}{2}\lim_{c\to\frac{\pi}{4}-}\left(\ln|\cos 2c|\right) = \infty$$

since $\cos(2c) \to 0+$ *as* $c \to (\pi/4)-$.

$$(d) \quad \int_{0+}^{\infty} \frac{1}{x^{1/4}+x+x^{2}}\,dx$$

This integral converges because, by definition:

$$\int_{0+}^{\infty} \frac{1}{x^{\frac{1}{4}} + x + x^2}\, dx = \int_{0+}^{1} \frac{1}{x^{\frac{1}{4}} + x + x^2}\, dx + \int_{1}^{\infty} \frac{1}{x^{\frac{1}{4}} + x + x^2}\, dx$$

and each of these two integrals converges (by virtue of the Comparison Test):

For $0 < x \leq 1$:

$$0 < \frac{1}{x^{\frac{1}{4}} + x + x^2} < \frac{1}{x^{\frac{1}{4}}}$$

and by the p-test for integrals of type II,

$$\int_{0+}^{1} \frac{1}{x^{\frac{1}{4}}}\, dx \text{ converges.}$$

For $x \geq 1$:

$$0 < \frac{1}{x^{\frac{1}{4}} + x + x^2} < \frac{1}{x^2}$$

and by the p-test for integrals of type I,

$$\int_{1}^{\infty} \frac{1}{x^2}\, dx \text{ converges.}$$

2. For each of the following *sequences*, determine *convergence* or *divergence*. In the case of convergence, find the *limit* of the sequence. Briefly justify each answer. *(Select any 7 of the 8 sequences. For extra credit, you may solve all eight.)*

(a) $$a_n = \frac{n + 2013 \ln n}{n!}$$

Since $n = o(n!)$ and $\ln n = o(n)$, $\{a_n\}$ converges and its limit is 0.

$$(b) \qquad b_n = 3^{-n} + \ln\left(\frac{n+1789}{n+1492}\right) + \left(1+\frac{4}{n}\right)^n$$

Using the fact that the limit of the sum of two convergent sequences is the sum of their limits, we have:

$$\lim b_n = \lim 3^{-n} + \lim \ln\left(\frac{n+1789}{n+1492}\right) + \lim\left(1+\frac{4}{n}\right)^n = 0 + \ln 1 + e^4 = e^4$$

Thus the sequence $\{b_n\}$ converges and its limit is e.

$$(c) \qquad c_n = \frac{\sin 4n}{n}$$

Since $-1 \le \sin(4n) \le 1$, we have:

$$-\frac{1}{n} \le \frac{\sin 4n}{n} \le \frac{1}{n}$$

Applying the Squeeze Theorem, we conclude that $\{c_n\}$ converges to 0.

$$(d) \qquad d_n = \frac{n^n}{n^{14}+1}$$

Observing that $d_n \ge n^n/(n^{14} + n^{14}) = \frac{1}{2} n^n/n^{14} \to \infty$ as $n \to \infty$, we conclude that d_n is unbounded, and thus divergent.

(e) $e_n = (-1)^n \cos(1/n)$

First note that, as n $\to \infty$, cos(1/n) $\to$ cos 0 = 1.

Thus for large n, e_n is approximately $(-1)^n$ which is a divergent sequence.

(f) $$f_n = \frac{(n^2+7n+1)^3(3n+77)(2n^5+n+4)^2}{1+\ln n + 5n^8(n+1)^9}$$

By selecting the dominant terms, we have:

$$\frac{(n^2+7n+1)^3(3n+77)(2n^5+n+4)^2}{1+\ln n + 5n^8(n+1)^9} \cong \frac{(n^2)^3(3n)(2n^5)^2}{5n^8(n)^9} = \frac{12n^{17}}{5n^{17}} = \frac{12}{5}$$

Hence we conclude that $\{f_n\}$ converges to 12/5.

(g) $g_n = \arctan(\ln(n))$

As n $\to \infty$, ln n $\to \infty$. and hence:

$$g_n = \arctan(\ln n) \to \frac{\pi}{2}$$

Thus the sequence $\{g_n\}$ converges and its limit is $\pi/2$.

(h) $$h_n = \sqrt{n^2+9n+31} - \sqrt{n^2+3n+5}$$

Rationalizing this expression:

$$h_n = \sqrt{n^2 + 9n + 31} - \sqrt{n^2 + 3n + 5} =$$

$$\left(\sqrt{n^2 + 9n + 31} - \sqrt{n^2 + 3n + 5}\right) \frac{\sqrt{n^2 + 9n + 31} + \sqrt{n^2 + 3n + 5}}{\sqrt{n^2 + 9n + 31} + \sqrt{n^2 + 3n + 5}} =$$

$$\frac{\left(n^2 + 9n + 31\right) - \left(n^2 + 3n + 5\right)}{\sqrt{n^2 + 9n + 31} + \sqrt{n^2 + 3n + 5}} = \frac{6n + 26}{\sqrt{n^2 + 9n + 31} + \sqrt{n^2 + 3n + 5}}$$

$$\cong \frac{6n}{n + n} = 3$$

Thus the sequence $\{h_n\}$ *converges and its limit is 3.*