SOLUTIONS: QUIZ V

1. Determine the *interval of convergence* of each of the following power series. Show your work! (You *need not* study end-point behavior.)

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{n^7\ 7^n}{\sqrt{n+2013}}(x+13)^n$

Applying the Ratio Test:

$$\left|\frac{\frac{(n+1)^7\ 7^{n+1}}{\sqrt{n+1+2013}}(x+13)^{n+1}}{\frac{n^7\ 7^n}{\sqrt{n+2013}}(x+13)^n}\right| = \sqrt{\frac{n+2013}{n+2014}}\left(\frac{n+1}{n}\right)^7 \frac{7^{n+1}}{7^n}\,|x+13| =$$

$$7\sqrt{\frac{n+2013}{n+2014}}\left(\frac{n+1}{n}\right)^7 |x+13| \to 7\,(1)\,(1)\,|x+13| = 7\,|x+13|$$

Thus the series converges absolutely when 7|x+13| < 1.

This is equivalent to |x+13| < 1/7. Hence the interval of convergence is (-13 – 1/7, -13 + 1/7) and the radius of convergence is 1/7.

(b) $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{1}{(n!)^2\,2^{2n}}(x-3)^{2n}$

Applying the Ratio Test:

$$\left|\frac{\frac{1}{((n+1)!)^2 2^{2n+2}}(x-3)^{2n+2}}{\frac{1}{(n!)^2 2^{2n}}(x-3)^{2n}}\right| = \left(\frac{n!}{(n+1)!}\right)^2 \frac{2^{2n}}{2^{2n+2}}\left|(x-3)^2\right| = \frac{(x-3)^2}{4(n+1)^2} \to 0 \text{ as } n \to \infty$$

Hence the series converges for all real numbers x. The interval of convergence is thus (-∞, ∞).

2. For each of the following numerical series, determine absolute convergence, conditional convergence or divergence. *Justify your answers.*

(a) $$\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln\left(1+\frac{1}{n}\right)}$$

Note that, as n → ∞:

$$|a_n| = \left|\frac{(-1)^n}{\ln\left(1+\frac{1}{n}\right)}\right| = \frac{1}{\ln\left(1+\frac{1}{n}\right)} \to \infty$$

Thus, since a_n does not converge to 0, the n^{th} term Test for Divergence implies that our series diverges.

(b) $$\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n\ln(n^4)}$$

Since n ln(n^4) = 4n ln n:

$$\sum_{n=2}^{\infty}\left|\frac{(-1)^{n+1}}{n\ln\left(n^4\right)}\right| = \frac{1}{4}\sum_{n=2}^{\infty}\frac{1}{n\ln n} \text{ diverges by the } p-\text{test}.$$

Hence our original series does not converge absolutely. However, it does converge conditionally because the Leibniz– Cauchy Theorem is applicable:

$$\frac{1}{4n\ln n} \to 0 \text{ as } n\to\infty \text{ and } \frac{1}{4(n+1)\ln(n+1)} < \frac{1}{4n\ln n} \quad \text{for all } n \geq 2$$

(c) $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{1+n^2}{(1+3n)^2}\right)^n$

Applying the Root Test we find that the series converges absolutely:

$$|a_n|^{\frac{1}{n}} = \left(\left(\frac{1+n^2}{(1+3n)^2}\right)^n\right)^{\frac{1}{n}} = \frac{1+n^2}{(1+3n)^2} \to \frac{1}{9} < 1$$

(d) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{(2n)!}{5^n(n!)^2}$

Applying the Ratio Test, we see that the series converges absolutely:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\dfrac{(2n+2)!}{5^{n+1}\left((n+1)!\right)^2}}{\dfrac{(2n)!}{5^n(n!)^2}} = \frac{(2n+2)!}{(2n)!}\ \frac{5^n}{5^{n+1}}\ \frac{(n!)^2}{\left((n+1)!\right)^2} =$$

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$$(2n+2)(2n+1)\left(\frac{1}{5}\right)\left(\frac{1}{n+1}\right)^2 = \frac{2}{5}\left(\frac{2n+1}{n+1}\right) \to \frac{4}{5} < 1$$

Extra Credit: For the following numerical series, determine absolute convergence, conditional convergence or divergence. *Justify your answer.*

$$\sum_{n=1}^{\infty} (-1)^n \left(\frac{\sin\left(\frac{1}{n}\right)}{n}\right)$$

We claim that the series converges absolutely.

Since

$$\frac{\sin\ h}{h} \to 1\ as\ h \to 0$$

we have

$$\frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} \to 1\ as\ n \to \infty$$

and thus, for large n:

$$0 < \frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} < 2$$

or equivalently:

$$0 < \sin\left(\frac{1}{n}\right) < \frac{2}{n}$$

So:

$$0 < \frac{\sin\left(\frac{1}{n}\right)}{n} < \frac{2/n}{n} = 2\left(\frac{1}{n^2}\right)$$

Using the Comparison Test, we find that our series converges absolutely since $\Sigma\, 1/n^2$ *converges (by the p-test).*

Alternatively, $\sin(1/n) \sim 1/n$. *So* $(\sin(1/n))/n \sim 1/n^2$. *Now use the Limit Comparison Theorem.*

If you disregard the very simplest cases, there is in all of mathematics not a single infinite series whose sum has been rigorously determined. In other words, the most important parts of mathematics stand without a foundation.

- Niels H. Abel (1802 - 1829)