SOLUTIONS: QUIZ VI

1. Find the 5^{th} order Taylor polynomial of **f(x) = ln x** centered at x = 3.

Solution:

Since f(x) = ln x, we find that:

$f'(x) = x^{-1}$

$f''(x) = -x^{-2}$

$f^{(3)}(x) = 2!\, x^{-3}$

$f^{(4)}(x) = -3!\, x^{-4}$

$f^{(5)}(x) = 4!\, x^{-5}$

Hence $f(3) = \ln 3$, $f'(3) = 3^{-1}$, $f''(3) = -3^{-2}$, $f^{(3)}(3) = 2!\, 3^{-3}$, $f^{(4)}(3) = -3!\, 3^{-4}$ *and* $f^{(5)}(3) = 4!\, 3^{-5}$.

So the 5^{th} *order Taylor polynomial of ln x centered at x = 3 is:*

$$P_5(x) = \ln 3 + \frac{1}{3(1!)}(x-3) - \frac{1}{3^2(2)}(x-3)^2 + \frac{1}{3^3(3)}(x-3)^3 - \frac{1}{3^4(4)}(x-3)^4$$

$$+ \frac{1}{3^5(5)}(x-3)^5$$

2. Without differentiating, find a Maclaurin polynomial for the function **h(x) = arctan(4x)** that has exactly *four* non-zero terms.

Hint: Begin with the expansion of 1/(1 – t) and then replace *t* by $-t^2$.

Solution:

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + t^4 + \ldots$$

Replacing t by t^2*:*

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + t^8 - \ldots$$

Integration produces:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots$$

Thus, the first four non-zero terms of the Maclaurin expansion are:

$$4x - \frac{4^3}{3}x^3 + \frac{4^5}{5}x^5 - \frac{4^7}{7}x^7$$

3. Find the 2^{nd} order Maclaurin polynomial of **G(x) = sec 3x.**

Solution:

Since G(x) = sec(3x), we have:

$G'(x) = 3\sec(3x)\tan(3x)$

$G''(x) = 3\{\sec(3x)\sec^2(3x)\,3 + \tan(3x)\,3\sec(3x)\tan(3x)\} = 9\{\sec^3(3x) + \sec(3x)\tan^2(3x)\}$

Hence $G(0) = 1$, $G'(0) = 0$, and $G''(0) = 9$.

So the 2^{nd} order Maclaurin polynomial of sec(3x) is

$$P_2(x) = 1 + (9/2!)\,x^2 = 1 + (9/2)\,x^2$$

4. Find the 4^{th} order Taylor polynomial centered at **x = 1** of the function **$F(x) = (x - 4)^4$**.

Solution:

Note that $F(1) = 81$, $F'(1) = -108$, $F''(1) = 108$, $F'''(1) = -72$ and $F^{(4)}(1)=24$.

Thus the 4^{th} degree Taylor polynomial is:

$$81 - \frac{108}{1!}(x-1) + \frac{108}{2!}(x-1)^2 - \frac{72}{3!}(x-1)^3 + \frac{24}{4!}(x-1)^4 =$$

$$81 - 108(x-1) + 54(x-1)^2 - 12(x-1)^3 + (x-1)^4$$

5. Suppose that the Maclaurin series of **g(x) = arcsinh(x)** is given by:

$$\sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1}$$

Find $g^{(5)}(0)$.

Solution:

We know that the coefficient of x^5 in the Maclaurin series expansion is $g^{(5)}(0)/5!$

Now the exponent is 5 when $2n + 1 = 5$. That is, when $n = 2$.

*So the coefficient of x^5 in the given series expansion is $(-1)^2(4!)/\{4^2)\ (2!)^2(5)\}$ which reduces to 3/40. Setting $3/40 = g^{(5)}(0)/5!$, we obtain: $g^{(5)}(0) = 3(5!)/40 =$ **9***

EXTRA CREDIT

Express the following integral as an infinite series.

$$\int_0^x \frac{1-\cos t}{t}\, dt$$

Solution:

$$\int_0^x \frac{1-\cos t}{t}\, dt = \int_0^x \frac{1-(1-\frac{t^2}{2!}+\frac{t^4}{4!}-...)}{t}\, dt = \int_0^x \frac{\frac{t^2}{2!}-\frac{t^4}{4!}+\frac{t^6}{6!}-...}{t}\, dt =$$

$$\int_0^x \left(\frac{t}{2!}-\frac{t^3}{4!}+\frac{t^5}{6!}-...\right) dt = \frac{x^2}{2(2!)}-\frac{x^4}{4(4!)}+\frac{x^6}{6(6!)}-...$$