MATH 162 SOLUTIONS: QUIZ VII

1. *Without using* l'Hôpital's rule, calculate the following limit. Show your work!

$$\lim_{t\to 0} \frac{t^2 e^{t^2} + t^2 - 2e^{t^2} + 2}{t^6}$$

Solution:

Using Maclaurin series:

$$\frac{t^2 e^{t^2} + t^2 - 2e^{t^2} + 2}{t^6} = \frac{t^2\left(1 + \frac{t^2}{1!} + \frac{t^4}{2!} + \ldots\right) + t^2 - 2\left(1 + \frac{t^2}{1!} + \frac{t^4}{2!} + \ldots\right) + 2}{t^6}$$

$$= \frac{\left(t^2 + \frac{t^4}{1!} + \frac{t^6}{2!} + \ldots\right) + t^2 - \left(2 + \frac{2t^2}{1!} + \frac{2t^4}{2!} + \frac{2t^6}{3!} + \ldots\right) + 2}{t^6} =$$

$$\frac{\frac{t^6}{2!} - \frac{2t^6}{3!} + O(t^8)}{t^6} = \frac{\frac{t^6}{6} + O(t^8)}{t^6} \to \frac{1}{6} \ as\ t \to 0$$

2. Let G(x) = x^8 sin (3x). Using an appropriate Maclaurin series, compute $G^{(2013)}(0)$. *(Do not try to simplify your answer.)*

Beginning with the Maclaurin series for sin t and then replacing t by 3x:

$$\sin t = \frac{t}{1!} - \frac{t^3}{3!} + \frac{t^5}{5!} - \ldots + (-1)^n \frac{t^{2n+1}}{(2n+1)!} + \ldots$$

$$\sin 3x = 3\frac{x}{1!} - 3^3 \frac{x^3}{3!} + 3^5 \frac{x^5}{5!} - \ldots + (-1)^n 3^{2n+1} \frac{x^{2n+1}}{(2n+1)!} + \ldots$$

Now, multiplying by x^8 yields:

$$G(x) = x^8 \sin 3x = 3\frac{x^9}{1!} - 3^3 \frac{x^{11}}{3!} + 3^5 \frac{x^{13}}{5!} - \ldots + (-1)^n 3^{2n+1} \frac{x^{2n+9}}{(2n+1)!} + .$$

Now, the general Maclaurin series of G(x) is:

$$G(x) = G(0) + \frac{G'(0)}{1!} x + \ldots + \frac{G^{(k)}(0)}{k!} x^k + \ldots$$

Thus the coefficient of x^{2013} is $G^{(2013)}(0) / 2013!$

Now the series for x^8 sin (3x) has coefficient of x^{2013} occur when 2n+9=2013, that is, when n =1002. Thus this coefficient is:

$(-1)^{1002} \cdot 3^{2005} / 2005! = 3^{2005} / 2005!$

Equating $G^{(2013)}(0) / 2013!$ with $3^{2005} / 2005!$, we find that:

*$G^{(2013)}(0)$ = **3^{2005} (2013!)/ (2005!)***

3. (a) Express $\frac{11-i}{12+i}$ as a complex number of the form a + bi.

Solution:

$$\frac{11-i}{12+i}=\left(\frac{11-i}{12+i}\right)\left(\frac{12-i}{12-i}\right)=\frac{131-23i}{145}=\frac{131}{145}-\frac{23}{145}i$$

(b) By expressing -1 as an appropriate complex power of *e*, calculate the four fourth roots of -1.

Since $-1 = e^{\pi i} = e^{3\pi i} = e^{5\pi i} = e^{7\pi i}$, *the four root of -1 are:*

$$e^{\frac{\pi}{4}i}=\cos\frac{\pi}{4}+i\ \sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i,$$

$$e^{\frac{3\pi}{4}i}=\cos\frac{3\pi}{4}+i\ \sin\frac{3\pi}{4}=-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i,$$

$$e^{\frac{5\pi}{4}i}=\cos\frac{5\pi}{4}+i\ \sin\frac{5\pi}{4}=-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i,$$

$$e^{\frac{7\pi}{4}i}=\cos\frac{7\pi}{4}+i\ \sin\frac{7\pi}{4}=\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i$$

(c) Express as a number in the form a + bi.

$$\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)^{i}$$

Using Euler's formula:

$$\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)^i=\left(e^{\frac{\pi}{4}i}\right)^i=e^{\frac{\pi}{4}i^2}=e^{-\frac{\pi}{4}}$$

4. Using Euler's identity, express cos(5x) in terms of cos x and sin x.

Solution:

Since $e^{xi} = \cos x + i \sin x$, *we have:*

$e^{5xi} = (\cos x + i \sin x)^5 = (\cos x)^5 + 5(\cos x)^4\, i \sin x + 10 (\cos x)^3\, i^2 (\sin x)^2 + 10 (\cos x)^2\, i^3 (\sin x)^3 + 5 \cos x\, (i^4) (\sin x)^4 + i^5 (\sin x)^5 = (\cos x)^5 + 5 (\cos x)^4 (\sin x)\, i - 10 (\cos x)^3 (\sin x)^2 - 10 (\cos x)^2 (\sin x)^3\, i + 5\cos x\, (\sin x)^4 + (\sin x)^5\, i$

But Euler's formula tells us that $e^{5xi} = \cos(5x) + i \sin(5x)$.

Thus $\cos(5x) = \cos^5 x - 10 \cos^3 x\, \sin^2 x + 5 \cos x \sin^4 x$

5. Suppose that the 4th degree Maclaurin polynomial of f(x) is $1 + 2x - 3x^2 + x^3 - 3x^4$ and that the 4th degree Maclaurin polynomial of g(x) is $1 - x + x^2 - x^3 - 5x^4$.

(a) Find the *first four non-zero* terms of the Maclaurin series of fg.

Answer: $1 + x - 4x^2 + 5x^3$

(b) Find the *first four non-zero* terms of the Maclaurin series of f/g.

Answer: $1 + 3x - x^2 - 2x^3$

EXTRA CREDIT:

Find the *first 4 non-zero terms* in the Maclaurin series expansion of arcsin (x^3). (*Hint:* Begin by calculating the first few terms of the Maclaurin (or binomial) series for $(1 – t)^{-1/2.}$)

Solution:

Using the definition of the Maclaurin series, or the binomial expansion, we obtain:

$$\frac{1}{\sqrt{1-t}} = (1-t)^{-\frac{1}{2}} = 1 + \frac{t}{2} + \frac{3t^2}{8} + \frac{5t^3}{16} + \ldots$$

Replacing t by t^2 yields:

$$\frac{1}{\sqrt{1-t^2}} = 1 + \frac{t^2}{2} + \frac{3t^4}{8} + \frac{5t^6}{16} + \ldots$$

Integration produces:

$$\arcsin x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} - \ldots$$

Thus:

$$\arcsin(x^3) = x^3 + \frac{1}{6}x^9 + \frac{3}{40}x^{15} + \frac{5}{112}x^{21} + \ldots$$