MATH 162 SOLUTIONS: TEST I

1. Consider the region bounded by the curves y = 3x + 4 and y = x^2. Find the volume of the solid generated by revolving this region about the line x = 4. Express your answer as a Riemann integral.

Using shells, we obtain:

$$V = 2\pi \int_{-1}^{4} (4 - x)(3x + 4 - x^2)\, dx$$

2. Evaluate $\int x \cosh(3x)\, dx$

Using integration by parts, we let f(x) = x and g′(x) = cosh 3x. Then f ′(x) = 1 and g(x) = (sinh 3x)/3. Thus

$$\int x \cosh(3x)\, dx = \frac{x \sinh(3x)}{3} - \int \frac{\sinh(3x)}{3}\, dx = \frac{x \sinh(3x)}{3} - \frac{\cosh(3x)}{9} + C$$

3. Evaluate $\int e^x \ln(e^x + 2013)\, dx$

Let us substitute u = e^x + 2013. Thus du = e^x dx, and so:

$$\int e^x \ln(e^x + 2013)\, dx = \int \ln u\, du = u \ln u - u + C = (e^x + 2013)\ln(e^x + 2013) - e^x + 2013 +$$

$$(e^x + 2013)\ln(e^x + 2013) - e^x + C$$

4. For each of the following statements answer *True* or *False*. Briefly justify each answer!

(a) $x^3 \ln x + x + 1 = o(x^4)$

True since $(x^3 \ln x + x + 1) / x^4 \to 0$ *as* $x \to \infty$.

(b) $\sinh x = O(\cosh x)$

True since $\sinh x / \cosh x = (e^x - e^{-x})/(e^x + e^{-x}) \to 1$ *as* $x \to \infty$.

(c) $\dfrac{3x^3(x^2+1)^5 + 5x\ln x + 99}{x^5 + 5x^3 + x + 2012} = o(x^8)$

False since:

$$\frac{\dfrac{3x^3(x^2+1)^5 + 5x\ln x + 99}{x^5 + 5x^3 + x + 2010}}{x^8} = \frac{3x^3(x^2+1)^5 + 5x\ln x + 99}{x^{13} + 5x^{11} + x^9 + 2010x^8} \to 3 \text{ as } x \to \infty$$

5. Consider the region in the first quadrant bounded by the curve $y = \cos x$, $0 \le x \le \pi/2$, and the x and y-axes. This region is rotated about the axis $y = 9$. Express the volume of this solid of revolution as a Riemann integral.

We will use washers to solve this problem. Fix x between 0 and π/2. The inner radius is 9 – cos x; the outer radius is 9. Hence:

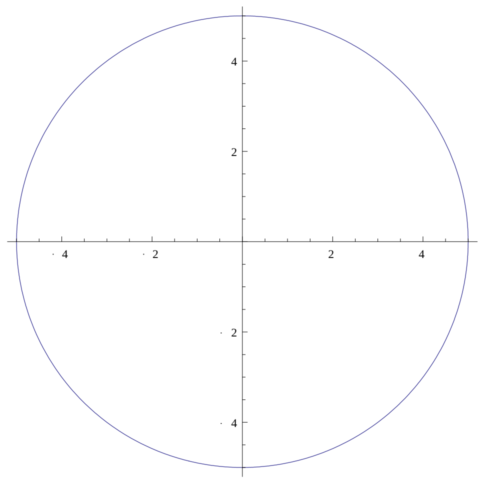
$$V = \pi \int_0^{\pi/2} \left(9^2 - (9 - \cos x)^2\right) dx$$

6. Evaluate $\int \arcsin x \, dx$

Integration by parts: Let $f(x) = \arcsin x$ and $g'(x) = 1$. Then $f'(x) = 1/(1-x^2)^{1/2}$ and $g(x) = x$. So

$$\int \arcsin x\ dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}}\, dx = x \arcsin x + \sqrt{1-x^2} + C$$

7. The base of a solid is a disk of radius 5. Each cross section cut by a plane perpendicular to a diameter is an isosceles right triangle with hypotenuse on the base. Express the volume of the solid as a Riemann integral. You need not evaluate the integral.



The equation of this circle is $x^2 + y^2 = 25$. Let us assume that the diameter referred to in the question lies on the x-axis. Then, taking a typical slice at x (in the interval [-5, 5], with thickness Δx, the volume of the corresponding slice (an isosceles right triangle with hypotenuse $2y = 2\,\mathrm{Sqrt}(25 - x^2)$ is given by

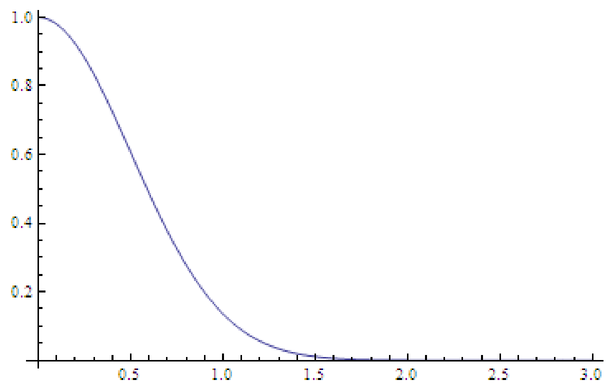
$\Delta V = \frac{1}{2}\, y\, (2y)\, \Delta x = (25 - x^2)\, \Delta x$. Thus:

$$V = \int_{-5}^{5} (25 - x^2)\, dx$$

8. Let *S* be the surface of revolution obtained by rotating the curve

$$y = e^{-2x^2}, \quad 0 \le x \le 3,$$

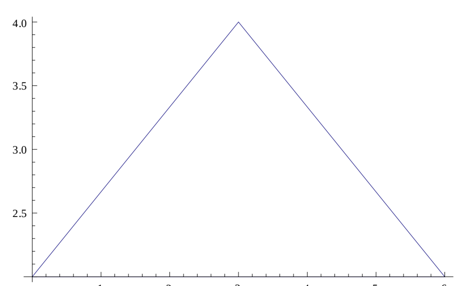
about the line x = 9. Find a Riemann integral that expresses the *surface area* of this region. (Do not evaluate the integral.)



$$S = \int_{x=0}^{3} 2\pi(9-x)\,ds = \int_{0}^{3} 2\pi(9-x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\;dx \;=$$

$$\int_{0}^{3} 2\pi(9-x)\sqrt{1+\left(-4xe^{-2x^2}\right)^2}\;dx = \int_{0}^{3} 2\pi(9-x)\sqrt{1+16x^2e^{-4x^2}}\;dx$$

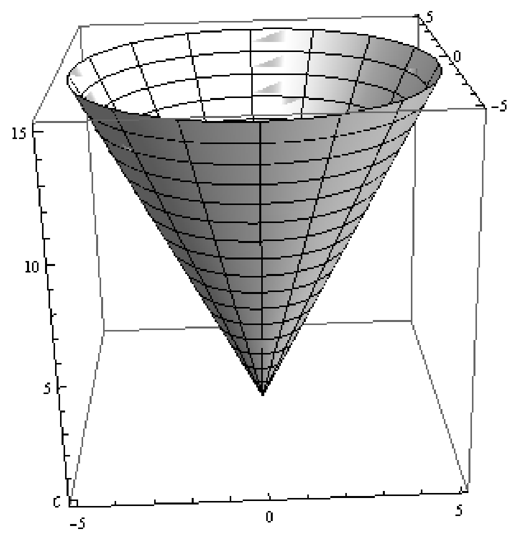
9. Consider the triangle with vertices (0, 2), (6, 2), (3, 4). This triangle is rotated about the axis y = -3. Express the volume of this solid of revolution as a Riemann integral. Do not evaluate.



The equations of the two non-horizontal sides are: y = (2/3)x + 2 and y = (-2/3)x + 6.
Solving for x, we obtain: x = (3/2) (y – 2) and x = -(3/2) (y – 6), respectively.
Using shells, the radius of the shell at y is y – (-3) = y + 3 and the length of the shell is -(3/2) (y – 6) – ((3/2) (y – 2)) = 12 – 3y. Hence:

$$V = \int_2^6 2\pi\left(y-(-3)\right)\left(12-3y\right)dy = 6\pi\int_2^6 \left(y+3\right)\left(4-y\right)dy$$

10. A conical tank with height 25 meters and radius 5 meters is filled with a fluid of density ρ kilograms per cubic meter. How much work must be done to pump all the fluid over the top rim of the tank? Do not evaluate the integral.



$$W = \int_0^{25} \rho\pi\left(\frac{y}{5}\right)^2 \left(25-y\right)dy \quad joules$$

11. Assume that *m* and *n* are positive integers. Using integration by parts, derive the following reduction formula:

$$\int x^m (\ln x)^n dx = \frac{x^{m+1}(\ln x)^n}{m+1} - \frac{n}{m+1}\int x^m (\ln x)^{n-1} dx$$

Let f(x) = (ln x)^n and g′(x) = x^m. Thus f′(x) = n(ln x)$^{n-1}$(1/x) and g(x) = x^{m+1}/(m+1). Thus:

$$\int x^m (\ln x)^n dx = f(x)g(x) - \int f'(x)g(x)dx = (\ln x)^n \frac{x^{m+1}}{m+1} - \int \frac{n(\ln x)^{n-1}}{x}\frac{x^{m+1}}{m+1}dx =$$

$$\frac{x^{m+1}(\ln x)^n}{m+1} - \frac{n}{m+1}\int x^m (\ln x)^{n-1} dx$$

12. The curve given by $x(t) = 1 + t\ln t$ *and* $y(t) = t + e^t$ *for* $1 \le t \le 5$ is rotated about the y-axis. Compute the *area* of the generated surface. Do not evaluate your integral.

$$S = \int_{t=1}^{5} 2\pi\, x(t)\ ds = 2\pi\int_{1}^{5} (1 + t\ln t)\sqrt{(1+\ln t)^2 + (1+e^t)^2}\ dt$$

EXTRA CREDIT: Evaluate the following integral:

$$\int \frac{\ln x}{\sqrt{x}}\, dx$$

Using the integration by parts formula:

Let f(x) = ln x and g′(x) = x$^{-1/2}$. Then f′(x)=1/x and g(x) =2 x$^{1/2}$. Hence:

$$\int \frac{\ln x}{\sqrt{x}}\, dx = 2x^{\frac{1}{2}} \ln x - \int \frac{2x^{\frac{1}{2}}}{x}\, dx = 2x^{\frac{1}{2}} \ln x - 2\int x^{-\frac{1}{2}}\, dx =$$

$$2\sqrt{x}\ln x - 4\sqrt{x} + C$$