MATH 162 SOLUTIONS: TEST II

PART I *(Answer all four problems.)*

1. Compute the value of the following improper integral:

$$\int_{0+}^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}}\, dx$$

Solution:

$$\int_{0+}^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}}\, dx = \lim_{c \to 0+} \int_{c}^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}}\, dx = \lim_{c \to 0+} 2\sqrt{\sin x}\Big|_{c}^{\pi/2} =$$

$$2 \lim_{c \to 0+} \left(\sqrt{\sin \frac{\pi}{2}} - \sqrt{\sin c} \right) = 2$$

2. Consider the following recursively defined sequence:

$c_1 = 1,$

$$c_{n+1} = \frac{1}{c_n} + 3 \quad \text{for } n \geq 1$$

(a) Find the values of c_2, c_3 and c_4.

Solution:

Setting n = 1:

$$c_2 = \frac{1}{c_1} + 3 = 1 + 3 = 4$$

Setting n = 2:

$$c_3 = \frac{1}{c_2} + 3 = \frac{1}{4} + 3 = \frac{13}{4} \cong 3.25$$

Setting n = 3:

$$c_4 = \frac{1}{c_3} + 3 = \frac{4}{13} + 3 = \frac{43}{13} \cong 3.307$$

(b) Assuming that the limit of c_n as $n \to \infty$ exists, find its exact value.

Solution:

Assume that $L = \lim c_n$ *exists. Then:*

$$\lim c_{n+1} = \lim\left(\frac{1}{c_n} + 3\right)$$

and so:

$$L = \frac{1}{L} + 3$$

Multiplying both sides by L yields: $L^2 = 1 + 3L$. *So:* $L^2 - 3L - 1 = 0$. *Using the quadratic formula:*

$$L = \frac{3 \pm \sqrt{13}}{2} \cong 3.3027,\ -0.3027$$

We reject the negative root, since $c_1 > 0$ *and all subsequent terms of the sequence are also positive (reasoning inductively).*

Thus, if lim c_n exists, this limit must be approximately 3.3027.

3. Determine *convergence* or *divergence* of the following improper integral.

Justify your answer:

$$\int_0^{13-} \frac{1}{13-x}\,dx$$

Solution:

Since $\int \frac{1}{13-x}\,dx = -\ln|13-x| + C$ *we have:*

$$\int_0^{13-} \frac{1}{13-x}\,dx = \lim_{c\to 13-} \int_0^{c} \frac{1}{13-x}\,dx = \lim_{c\to 13-} (-1)\ \ln(13-x)\Big|_0^c =$$

$$-\lim_{c\to 13-} (\ln(13-c) - \ln 13) = \infty$$

Thus our integral diverges.

4. Determine *convergence* or *divergence* of the following improper integral.

Justify your answer:

$$\int_{0+}^{5} \frac{1}{\sqrt{x^3+x^4}}\,dx$$

Solution:

Since $x^3 > x^4$ when $0 < x < 1$:

$$\frac{1}{\sqrt{x^3 + x^4}} \geq \frac{1}{\sqrt{x^3 + x^3}} = \frac{1}{\sqrt{2}}\left(\frac{1}{x^{1.5}}\right)$$

Now we know that $\int_{0+}^{1} \frac{1}{x^{1.5}}\,dx$ *diverges, from the p-test for integrals of the second kind. Thus, invoking the Comparison Test, we find that*

$\int_{0+}^{1} \frac{1}{\sqrt{x^3 + x^4}}\,dx$ *diverges.*

PART II *Select any 5 of the following 6 sequences.* For each selected sequence, determine *convergence* or *divergence*. Briefly justify each answer. *In the case of convergence, find the limit.* Calculator results will not earn full credit. (You may answer all 6 to earn extra credit.)

1. $a_n = n\,\sin\left(\frac{1}{13n}\right)$

Solution: Let $h = 1/(13n)$. Then $n = 1/(13h)$ and as $n \to \infty$, $h \to 0$. Hence:

$$a_n = n\,\sin\left(\frac{1}{13n}\right) = \frac{1}{13h}\sin h = \frac{1}{13}\left(\frac{\sin h}{h}\right) \to \frac{1}{13} \;\; as \;\; h \to 0$$

Hence the sequence a_n converges to 1/13.

2. $$b_n = \left(1 + \frac{13}{n}\right)^{2n}$$

Solution: Note that:

$$b_n = \left(1 + \frac{13}{n}\right)^{2n} = \left(\left(1 + \frac{13}{n}\right)^{n}\right)^{2} \rightarrow \left(e^{13}\right)^{2} = e^{26}$$

Hence the sequence b_n converges to e^{26}.

3. $$c_n = \sqrt{n^4 + 19n^2 + 5} - \sqrt{n^4 - 7n^2 + n + 13}$$

Rationalizing the "numerator" yields:

$$c_n = \left(\sqrt{n^4 + 19n^2 + 5} - \sqrt{n^4 - 7n^2 + n + 13}\right)\left(\frac{\sqrt{n^4 + 19n^2 + 5} + \sqrt{n^4 - 7n^2 + n + 13}}{\sqrt{n^4 + 19n^2 + 5} + \sqrt{n^4 - 7n^2 + n + 13}}\right) =$$

$$\frac{26n^2 - n - 8}{\sqrt{n^4 + 19n^2 + 5} + \sqrt{n^4 - 7n^2 + n + 13}} \cong \frac{26n^2}{\sqrt{n^4} + \sqrt{n^4}} = 13$$

Hence the sequence c_n converges to 13.

4. $$d_n = \ln(35n + 2013) - \ln(5n - 11) + \arctan n$$

Solution:

$$d_n = \ln(35n + 2013) - \ln(5n - 11) + \arctan n =$$

$$\ln \frac{35n + 2013}{5n - 11} + \arctan n \to \ln 7 + \frac{\pi}{2}$$

5. $e_n = \int_{\frac{1}{n}}^{1} \frac{1}{\sqrt{t}} dt$

Solution:

$$e_n = \int_{\frac{1}{n}}^{1} \frac{1}{\sqrt{t}} dt = 2\sqrt{t}\Big|_{\frac{1}{n}}^{1} = 2\left(1 - \frac{1}{\sqrt{n}}\right) \to 2$$

6. $f_n = \sqrt{13 + 44\frac{\ln n}{n} + 77\cos\left(\frac{1}{n}\right) + 79\tanh n}$

Solution:

$$f_n = \sqrt{13 + 44\frac{\ln n}{n} + 77\cos\left(\frac{1}{n}\right) + 79\tanh n} \quad \to$$

$$\sqrt{13 + 44(0) + 77(1) + 79(1)} = \sqrt{169} = 13$$

PART III Select any 5 of the following 6 series. For each selected series, determine *convergence* or *divergence*. Justify each answer. (You may answer all 6 to earn extra credit.)

1. $\sum_{n=1}^{\infty} \left(\frac{1}{n+12} - \frac{1}{n+13}\right)$

Solution: This series is telescoping. Consider the sequence of partial sums:

$$s_1 = \frac{1}{13} - \frac{1}{14}$$

$$s_2 = \left(\frac{1}{13} - \frac{1}{14}\right) + \left(\frac{1}{14} - \frac{1}{15}\right) = \frac{1}{13} - \frac{1}{15}$$

$$s_3 = \left(\frac{1}{13} - \frac{1}{14}\right) + \left(\frac{1}{14} - \frac{1}{15}\right) + \left(\frac{1}{15} - \frac{1}{16}\right) = \frac{1}{13} - \frac{1}{16}$$

from which we infer that $s_n = 1/13 - 1/(13+n) \to 0$ *as* $n \to \infty$. *Thus the sum of our series is 1/13.*

2. $\sum_{n=1}^{\infty} \frac{7n^2}{4^{2n}}$

Solution: Applying the ratio test

$$\frac{a_{n+1}}{a_n} = \frac{\frac{7(n+1)^2}{4^{2(n+1)}}}{\frac{7n^2}{4^{2n}}} = \frac{4^{2n}}{4^{2n+2}} \frac{7}{7} \frac{(n+1)^2}{n^2} =$$

$$\frac{1}{16}\left(\frac{n+1}{n}\right)^2 \to \frac{1}{16} < 1$$

we find that the series converges since r < 1.

3. $\sum_{n=1}^{\infty} \left(\frac{n+4}{n}\right)^n$

Solution: Since $\left(\frac{n+4}{n}\right)^n = \left(1+\frac{4}{n}\right)^n \to e^4 \neq 0,$ *we apply the* n^{th} *Term Test for Divergence to conclude that our series diverges.*

4. 13.13131313…

Solution: Observe that 13.13131313... = 13(1 + 10^{-2} + 10^{-4} + 10^{-6} + ...).

Ignoring the factor of 13 for the moment, we have a geometric series with ratio R

= 0.01. Hence $13.13131313... = 13\left(\frac{1}{1-R}\right) = 13\left(\frac{1}{1-0.01}\right) = \frac{13}{0.99} = \frac{1300}{99}$

5. $\sum_{n=1}^{\infty} \frac{n^3}{\sqrt{1+n+n^9}}$

Solution: Using the Comparison Test:

$$0 < \frac{n^3}{\sqrt{1+n+n^9}} < \frac{n^3}{\sqrt{n^9}} = \frac{1}{n^{\frac{3}{2}}}$$

Using the p-test for the larger series, we see that our series converges.

6. $\sum_{n=2}^{\infty} \frac{1+\sin^4(n)}{n\ln(n^{13})}$

Solution: Consider the following inequality:

$$\frac{1+\sin^4(n)}{n\ln(n^{13})} \geq \frac{1}{13n\ln n} > 0$$

Using the p-test, we see that the smaller series diverges and hence our series diverges as well.

PART IV Select any five of the following six problems. You may answer all six for extra credit. For each improper integral below, determine convergence or divergence. *Justify each answer!*

$(A)\quad \int_0^\infty \cos^2 x\ dx$

Solution:

Since

$$\int_0^{2\pi} \cos^2 x\ dx = \pi$$

we see that

$$\int_0^{2n\pi} \cos^2 x\ dx = n\pi \to \infty$$

Hence our integral diverges.

$(B)\quad \int_1^\infty \frac{2013+\ln x}{x^3}\ dx$

Solution:

Since $\ln x < x$ *, we have:*

$$0 < \frac{2013+\ln x}{x^3} < \frac{2014\ln x}{x^3} < \frac{2014x}{x^3} = 2014\,\frac{1}{x^2}$$

Thus, invoking both the p-test and the Comparison Test, our original integral converges.

$(C)\quad \int_0^{\infty} \frac{1+xe^{2x}}{1+e^{4x}}\,dx$

Solution:

Observe that

$$0 < \frac{1+xe^{2x}}{1+e^{4x}} < \frac{xe^{2x}+xe^{2x}}{e^{4x}} = \frac{2xe^{2x}}{e^{4x}} = \frac{2x}{e^{2x}} < \frac{2e^{x}}{e^{2x}} = \frac{2}{e^{x}}$$

Now $\int_0^{\infty} e^{-x}\,dx$ *converges.*

Thus, invoking the Comparison Test, our original integral converges.

$(D)\quad \int_0^{\infty} \frac{9+2013x^{6}+13\sqrt{x}}{5+4\sqrt{x}+x^{8}}\,dx$

Solution:

Observe that

$$0 < \frac{9 + 2013x^6 + 13\sqrt{x}}{5 + 4\sqrt{x} + x^8} < \frac{9x^6 + 2013x^6 + 13x^6}{x^8} = 2035\frac{1}{x^2}$$

Thus, invoking the Comparison Test, our original integral converges.

(*E*) $\displaystyle\int_2^{\infty} \frac{4 + 3\cos^4(4x+1)}{x + 2013}\, dx$

Solution:

Observe that

$$\frac{4 + 3\cos^4(4x+1)}{x + 2013} > \frac{1}{x + 2013x} = \frac{1}{2014}\frac{1}{x} > 0$$

Thus, invoking the Comparison Test, our original integral diverges.

(*F*) $\displaystyle\int_0^{\pi/2-} \tan^2 x\, dx$ (*Hint:* Use a common identity.)

Solution:

$$\int_0^{\pi/2-} \tan^2 x\, dx = \lim_{b \to \pi/2-} \int_0^{b} \tan^2 x\, dx = \lim_{b \to \pi/2-} \int_0^{b} (\sec^2 x - 1)\, dx =$$

$$\lim_{b \to \pi/2-} (\tan x - x)\Big|_0^b = \lim_{b \to \pi/2-} \big((\tan b - b) - (\tan 0 - 0)\big) = \infty$$

Thus the original integral diverges.

PART V Select any 4 of the following 5 problems. You may answer all five for extra credit. For each numerical series below, determine *convergence* or *divergence*. Justify each answer.

1. $$\sum_{n=1}^{\infty} \frac{n^5 2^n (n!)^2}{(2n)!}$$

Solution: Applying the ratio test to this positive series:

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{(n+1)^5 2^{n+1}((n+1)!)^2}{(2n+2)!}}{\dfrac{n^5 2^n (n!)^2}{(2n)!}} = \frac{(n+1)^5}{n^5} \frac{2^{n+1}}{2^n} \frac{((n+1)!)^2}{(n!)^2} \frac{(2n)!}{(2n+2)!} =$$

$$\left(\frac{n+1}{n}\right)^5 (2) \left(\frac{(n+1)!}{n!}\right)^2 \frac{1}{(2n+2)(2n+1)} = 2\left(\frac{n+1}{n}\right)^5 \frac{(n+1)^2}{2(n+1)(2n+1)} =$$

$$\left(\frac{n+1}{n}\right)^5 \frac{(n+1)^2}{(n+1)(2n+1)} \to 1^5 \frac{1}{2} = \frac{1}{2} = r$$

Since r < 1, we conclude that our positive series converges.

2. $$\sum_{n=1}^{\infty} \frac{1}{\left(1+\frac{2}{n}\right)^{n^2}}$$

Solution: Applying the nth root test to this positive series:

$$(a_n)^{1/n} = \frac{1}{\left(1+\frac{2}{n}\right)^{n}} \rightarrow \frac{1}{e^2} < 1 = \rho$$

Since $\rho < 1$, we conclude that our positive series converges.

3. $$\sum_{n=1}^{\infty} \frac{n^5}{e^{n^2}}$$

Solution:

Applying the ratio test to this positive series:

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^5}{e^{(n+1)^2}}}{\frac{n^5}{e^{n^2}}} = \left(\frac{n+1}{n}\right)^5 \frac{e^{n^2}}{e^{(n+1)^2}} =$$

$$\left(\frac{n+1}{n}\right)^5 \frac{1}{e^{2n+1}} \rightarrow \left(1^5\right)(0) = 0$$

4. $\sum_{n=1}^{\infty}\left(\frac{n+3}{n+111}\right)^{33}$

Solution: Since

$$\left(\frac{n+3}{n+111}\right)^{33}=\left(\frac{1+3/n}{1+111/n}\right)^{33}\to 1^{33}=1,$$

We may invoke the n^th^ Term Test for Divergence to conclude that our original series diverges.

5. $\sum_{n=1}^{\infty}\frac{1}{n+4\ln n+\sqrt[5]{n}+2013}$

Solution:

Since $\frac{1}{n+4\ln n+\sqrt[5]{n}+2013}>\frac{1}{n+4n+n+2013n}>\frac{1}{2019n}>0$ *we can invoke the Comparison Test to conclude that our series diverges.*