SOLUTIONS: TEST III

Instructions: *Answer any 9 of the following 11 problems. You may answer more than 9 to obtain extra credit.*

1. Without using l'Hôpital's rule, find:

$$\lim_{x\to 0} \frac{e^{3x^2} - 1 - 3x^2 - \frac{9}{2}x^4 - \frac{9}{2}x^6}{\cos(x^4) - 1}$$

Solution:

Since

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + O(x^5)$$

it follows that:

$$e^{3x^2} = 1 + \frac{3x^2}{1!} + \frac{(3x^2)^2}{2!} + \frac{(3x^2)^3}{3!} + \frac{(3x^2)^4}{4!} + O(x^{10}) =$$

$$1 + 3x^2 + \frac{9x^4}{2} + \frac{27x^6}{6} + \frac{81x^8}{24} + O(x^{10}) =$$

$$1 + 3x^2 + \frac{9x^4}{2} + \frac{9x^6}{2} + \frac{27x^8}{8} + O(x^{10})$$

Since

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + O(x^6)$$

it follows that:

$$\cos\left(x^4\right) = 1 - \frac{x^8}{2!} + O(x^{16})$$

Hence:

$$\frac{e^{3x^2} - 1 - 3x^2 - \frac{9}{2}x^4 - \frac{9}{2}x^6}{\cos(x^4) - 1} =$$

$$\frac{1 + 3x^2 + \frac{9x^4}{2} + \frac{9x^6}{2} + \frac{27x^8}{8} + O(x^{10}) - 1 - 3x^2 - \frac{9}{2}x^4 - \frac{9}{2}x^6}{1 - \frac{x^8}{2!} + O(x^{16}) - 1} =$$

$$\frac{\frac{27x^8}{8} + O(x^{10})}{-\frac{x^8}{2!} + O(x^{16})} \to \frac{\frac{27}{8}}{-\frac{1}{2!}} = -\frac{27}{4}$$

2. Evaluate each of the following indefinite integrals.

(a) $$\int \frac{1}{26 + 10x + x^2}\, dx$$

Solution:

After completing the square, make the change of variable u = x + 5 (and thus du = dx):

$$\int \frac{1}{26+10x+x^2}\,dx = \int \frac{1}{(x+5)^2+1}\,dx =$$

$$\int \frac{1}{u^2+1}\,du = \arctan u + C = \arctan(x+5)+C$$

(b) $\displaystyle\int \frac{1}{(x^2+1)^{3/2}}\,dx$

Solution:

Let x = tan t. Then dx = sec^2 t dt. Hence:

$$\int \frac{1}{(x^2+1)^{3/2}}\,dx = \int \frac{\sec^2 t}{(\tan^2 t+1)^{3/2}}\,dt =$$

$$\int \frac{\sec^2 t}{(\sec^2 t)^{3/2}}\,dt = \int \cos t\;dt = \sin t + C =$$

$$\sin(\arctan x) + C = \frac{x}{\sqrt{x^2+1}} + C$$

3. Evaluate each of the following indefinite integrals. Show your work.

(a) $\displaystyle\int \left(\cos^3 x\right)\sqrt{\sin x}\;dx$

Solution:

$$\int \left(\cos^3 x\right)\sqrt{\sin x}\;\; dx = \int \left(1-\sin^2 x\right)\sqrt{\sin x}\;\; \cos x\, dx =$$

$$\int (\sin x)^{\frac{1}{2}}\;\; \cos x\, dx - \int (\sin x)^{\frac{5}{2}}\;\; \cos x\, dx = \frac{2}{3}(\sin x)^{\frac{3}{2}} - \frac{2}{7}(\sin x)^{\frac{7}{2}} + C$$

(b) $\int \sin(3x)\cos x\;\; dx$

Solution: Since sin(4x) = sin(3x+x) = sin 3x cos x + cos 3x sin x,
we obtain: sin 3x cox x = ½ (sin 4x + sin 2x). Thus:

$$\int \sin(3x)\cos x\;\; dx = \frac{1}{2}\int \sin(4x) + \sin(2x)\;\; dx =$$

$$-\frac{1}{8}\cos 4x - -\frac{1}{4}\cos 2x + C$$

(c) $\int \sin^4 x \cos^5 x\, dx$

Solution:

$$\int \sin^4 x \cos^5 x\, dx = \int \sin^4 x \cos^4 x \cos x\, dx =$$

$$\int \sin^4 x\left(\cos^2 x\right)^2\;\; \cos x\, dx = \int \sin^4 x\left(1-\sin^2 x\right)^2\;\; \cos x\, dx$$

Introduce the change of variable u = sin x (and thus du = cos x dx) to

obtain:

$$\int \sin^4 x\left(1-\sin^2 x\right)\cos x\,dx=\int u^4\left(1-u^2\right)^2\,du=$$

$$\int\left(u^4-2u^6+u^8\right)du=\frac{u^5}{5}-\frac{2u^7}{7}+\frac{u^9}{9}+C=$$

$$\frac{1}{5}\sin^5 x-\frac{2}{7}\sin^7 x+\frac{1}{9}\sin^9 x+C$$

4. Given y = G(x) below, calculate the value of $G^{(1313)}(0)$. (Express your answer in factorial form.)

$$G(x)=x^3\sinh(x^2)$$

Solution:

Beginning with the Maclaurin series for sinh t and then replacing t by x^2:

$$\sinh t=\frac{t}{1!}+\frac{t^3}{3!}+\frac{t^5}{5!}-\ldots+\frac{t^{2n+1}}{(2n+1)!}+\ldots$$

$$\sinh\left(x^2\right)=\frac{x^2}{1!}+\frac{x^6}{3!}+\frac{x^{10}}{5!}+\ldots+\frac{x^{4n+2}}{(2n+1)!}+\ldots$$

Now, multiplying by x^3 yields:

$$G(x)=x^3\sinh\left(x^2\right)=\frac{x^5}{1!}+\frac{x^9}{3!}+\frac{x^{13}}{5!}+\ldots+\frac{x^{4n+5}}{(2n+1)!}+\ldots$$

Now, the general Maclaurin series of G(x) is:

$$G(x) = G(0) + \frac{G'(0)}{1!}x + \ldots + \frac{G^{(k)}(0)}{k!}x^k + \ldots$$

Thus the coefficient of x^{1313} is $G^{(1313)}(0) / 1313!$

Now the series for $x^3 \sinh(x^2)$ has coefficient of x^{1313} occur when $4n + 5 = 1313$, that is, when $n = 327$ (and so $2n + 1 = 655$). Thus this coefficient is: $1 / 655!$

Equating $G^{(1313)}(0) / 1313!$ with $1 / 655!$, we find that:

$G^{(1313)}(0) = 1313! / 655!$

5. (a) Using Euler's formula, find a formula for sin(4x) in terms of sin x and cos x.

Solution:

Since $e^{xi} = \cos x + i \sin x$, we have:

$e^{4xi} = (\cos x + i \sin x)^4 = (\cos x)^4 + 4(\cos x)^3 i \sin x + 6 (\cos x)^2 i^2 (\sin x)^2 + 4 (\cos x) i^3 (\sin x)^3 + (i^4) (\sin x)^4 = (\cos x)^4 + 4 (\cos x)^3 (\sin x) i - 6 (\cos x)^2(\sin x)^2 - 4 (\cos x)(\sin x)^3 i + (\sin x)^4$

But Euler's formula tells us that $e^{4xi} = \cos(4x) + i \sin(4x)$.

Thus $\mathbf{\sin(4x) = 4 \cos^3 x \sin x - 4 \cos x \sin^3 x}$

(b) Using Euler's formula find all the solutions of the equation

$$z^4 = -i$$

Express each answer in the form a + bi.

Solution:

Since

$$-i = e^{\frac{3}{2}\pi i} = e^{\frac{3}{2}\pi i}e^{2\pi i} = e^{\frac{3}{2}\pi i}e^{4\pi i} = e^{\frac{3}{2}\pi i}e^{6\pi i}$$

we find that the four fourth roots of –i are:

$$e^{\frac{3\pi}{8}i} = \cos\frac{3\pi}{8} + i\,\sin\frac{3\pi}{8}$$

$$e^{\frac{7\pi}{8}i} = \cos\frac{7\pi}{8} + i\,\sin\frac{7\pi}{8}$$

$$e^{\frac{11\pi}{8}i} = \cos\frac{11\pi}{8} + i\,\sin\frac{11\pi}{8}$$

$$e^{\frac{15\pi}{8}i} = \cos\frac{15\pi}{8} + i\,\sin\frac{15\pi}{8}$$

6. By dividing power series, find the *first four non-zero* terms of the Maclaurin series of

$$\frac{e^{2x}}{1+\sin x}$$

Solution:

$$\frac{e^{2x}}{1+\sin x} = \frac{1+2x+\frac{2^2}{2!}x^2+\frac{2^3}{3!}x^3+\ldots}{1+(x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\ldots)} =$$

$$\frac{1+2x+2x^2+\frac{4}{3}x^3+\ldots}{1+x-\frac{1}{6}x^3+\ldots} = 1+x+x^2+\frac{1}{2}x^3+\ldots$$

7. Through an appropriate change of variables, convert each of the following to a trigonometric integral. *Do not evaluate.*

$$(a)\quad \int \frac{\sqrt{x^2-1}}{x^9}\,dx$$

Solution:

Let x = sec θ. Then ***dx = sec θ tan θ dθ*** *and so:*

$$\int \frac{\sqrt{x^2-1}}{x^9}\,dx = \int \frac{\sqrt{\sec^2\theta-1}}{\sec^9\theta}\sec\theta\tan\theta\;d\theta =$$

$$\int \frac{\tan\theta}{\sec^8\theta}\tan\theta\;d\theta = \int \frac{\tan^2\theta}{\sec^8\theta}\;d\theta$$

$$(b)\quad \int \frac{x^3}{\left(9+x^2\right)^{\frac{11}{2}}}\,dx$$

Solution:

Let x = 3 tan θ. Then ***dx = 3 sec² θ dθ*** *and so:*

$$\int \frac{x^3}{\left(9+x^2\right)^{\frac{11}{2}}}\,dx = \int \frac{\tan^3\theta}{\left(9+\tan^2\theta\right)^{\frac{11}{2}}}\left(3\sec^2\theta\right)d\theta =$$

$$3\int \frac{\tan^3\theta}{3^{11}\sec^{11}\theta}\sec^2\theta\ d\theta = \frac{1}{3^{10}}\int \frac{\tan^3\theta}{\sec^9\theta}\ d\theta$$

8. For each pair of integrals, determine which one is the *more difficult* to evaluate. (You need not evaluate any of these integrals.) *Briefly explain!*

$$(a)\quad \int \left(\sin^{2013}x\right)\left(\cos x\right)dx \quad or \quad \int \left(\sin^4 x\right)\left(\cos^4 x\right)dx$$

The first integral is easier to evaluate since it is of the form

$$\int u^{2013}\, du$$

The second integral requires that one begin with the double angle formulas for sine and cosine.

$$(b) \quad \int \left(\tan^{2013} x\right)\sec^2 x\, dx \quad or \quad \int \left(\tan^{2013} x\right)\sec x\, dx$$

The first integral is easier to evaluate since it is of the form

$$\int u^{2013}\, du$$

9. For each series below, determine *absolute convergence*, *conditional convergence* or *divergence*. Justify each answer.

$$(a) \quad \sum_{n=3}^{\infty} (-1)^n \frac{13}{n\sqrt{\ln n}}$$

Solution:

Notice that this series fails to converge absolutely, by the p-test. It does converge, however, due to Cauchy's test. Thus the series converges conditionally.

$$(b) \quad \sum_{k=1}^{\infty} (-1)^k \arctan(k^2)$$

Solution:

Since arctan(k^2) → π/2 as k → ∞, the series diverges by the n^{th}-term test for divergence.

$$(c)\ \sum_{n=1}^{\infty} (-1)^n \frac{(n!)^2}{(2n)!}$$

Applying the Ratio Test, we see that the series converges absolutely:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\dfrac{\left((n+1)!\right)^2}{(2(n+1))!}}{\dfrac{(n!)^2}{(2n)!}} = \frac{(2n)!}{(2n+2)!}\,\frac{\left((n+1)!\right)^2}{\left(n!\right)^2} =$$

$$\frac{1}{(2n+2)(2n+1)}\left(\frac{(n+1)!}{n!}\right)^2 = \frac{(n+1)^2}{(2n+2)(2n+1)} \to \frac{1}{4} < 1$$

10. For each power series below, determine the *interval of convergence*. Do not investigate the behavior of each power series at the endpoints.

$$(a)\ \sum_{n=1}^{\infty} \frac{n^{13}}{13^n}(x-13)^n$$

Using the ratio test:

$$\left| \frac{\frac{(n+1)^{13}}{13^{n+1}}(x-13)^{n+1}}{\frac{n^{13}}{13^n}(x-13)^n} \right| = \frac{1}{13}\frac{(n+1)^{13}}{n^{13}}|x-13| =$$

$$\frac{1}{13}\left(\frac{n+1}{n}\right)^{13}|x-13| \to \frac{1}{13}|x-13|$$

Thus the series converges absolutely for |x – 13| / 13 < 1. So the interval of convergence is (0, 26).

(b) $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}(x-4)^n$

Invoking the root test:

$$\left|\left(1+\frac{1}{n}\right)^{n^2}(x-4)^n\right|^{1/n} = \left(1+\frac{1}{n}\right)^n |x-4| \to e|x-4|$$

Thus the series converges absolutely for e |x – 4| < 1. So the interval of convergence is (4 – 1/e, 4 + 1/e).

11. For the power series below, determine the *interval of convergence*. Investigate *end point behavior*.

$(a)\ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+13}}(x-13)^n$

Solution:

Using the ratio test:

$$\left| \frac{\frac{1}{\sqrt{n+14}}(x-13)^{n+1}}{\frac{1}{\sqrt{n+13}}(x-13)^n} \right| = \frac{\sqrt{n+13}}{\sqrt{n+14}}|x-13| = \sqrt{\frac{n+13}{n+14}}|x-13| \to |x-13|$$

Thus the series converges absolutely for |x – 13| < 1. So the interval of convergence is (12, 14).

At x = 14, the series equals:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+13}}(14-13)^n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+13}}$$

which diverges (using the comparison test and the p-test).

At x = 12, the series equals:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+13}}(12-13)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+13}}$$

which converges conditionally (using Cauchy's test as well as the fact that it fails to converge absolutely).

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{13^n}{n^2} x^{2n}$

Solution:

Using the ratio test:

$$\left| \frac{\dfrac{13^{n+1}}{(n+1)^2} x^{2(n+1)}}{\dfrac{13^n}{n^2} x^{2n}} \right| = 13 \frac{n^2}{(n+1)^2} x^2 = 13\left(\frac{n}{n+1}\right)^2 x^2 \to 13x^2$$

Thus the series converges absolutely for $13x^2 < 1$. So the interval of convergence is $\left(-1/\sqrt{13},\ 1/\sqrt{13}\right)$.

At $x = 1/\sqrt{13}$,

$$\sum_{n=1}^{\infty} \frac{13^n}{n^2} x^{2n} = \sum_{n=1}^{\infty} \frac{13^n}{n^2} \left(\frac{1}{13}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

which converges absolutely (using the p-test).

At $x = -1/\sqrt{13}$,

$$\sum_{n=1}^{\infty} \frac{13^n}{n^2} x^{2n} = \sum_{n=1}^{\infty} \frac{13^n}{n^2} \left(\frac{-1}{\sqrt{13}}\right)^{2n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

which converges absolutely (using the p-test).

Extra Credit: Using a series representation of sin(3x), find constants *r* and *s* for which:

$$\lim_{x\to 0}\left(\frac{\sin(3x)}{x^3}+\frac{r}{x^2}+s\right)=0$$

Solution:

Since

$$\sin t = t-\frac{1}{3!}t^3+\frac{1}{5!}t^5-\ldots$$

we have:

$$\frac{\sin(3x)}{x^3}+\frac{r}{x^2}+s=\frac{\sin(3x)+rx+sx^3}{x^3}=$$

$$\frac{(3x-\frac{3^3}{3!}x^3+\frac{3^5}{5!}x^5-\ldots)+rx+sx^3}{x^3}=$$

$$\frac{(3+r)x+\left(s-\frac{3^3}{3!}\right)x^3+\frac{3^5}{5!}x^5-\ldots}{x^3}$$

If this limit equals 0, then $3 + r = 0$ *and* $s - 3^3/(3!) = 0$.

Hence $r = -3$ *and* $s = 9/2$.