WORKSHEET X

SEQUENCES

1. Explain precisely what it means for a sequence $\{a_n\}$ to *converge*. What does it mean to say that a sequence *diverges*? What is meant by the *limit* of a sequence?

2. Discuss the rules for convergence (and divergence) of a sum, difference, product, and quotient of two sequences. State the Squeeze Theorem for sequences. What can be said about an increasing sequence that is bounded above? Is every bounded sequence convergent? Is every convergent sequence bounded?

3. For each of the following sequences, $\{t_n\}$, determine convergence or divergence. If the sequence converges, are you able to find its limit?

(a) $a_n = (-1)^n$

(b) $t_n = \sin(\pi n/2)$

(c) $t_n = \cos(\pi/n)$

(d) $t_n = (\ln n) / n$

(e) $t_n = 3 \arctan(n^2)$

(f) $t_n = (\ln n) / (\ln \ln n)$

(g) $t_n = (1 + 1/n)^n$

(h) $t_n = (-1)^{n+1} n^2$

(i) $t_n = ½ + (-1)^n/2$

(j) $t_n = n! / (n+3)$

(k) $t_n = (\cosh n) / (\sinh n)$

(l) $t_n = 1/2 + 1/3 + 1/4 + ... + 1/n$

(m) $t_n = (n^4 – 3n^2 + n^5 + 13) / (n^3 \ln n – 5n^5 + \ln(1+n^6) – 99)$

(n) $t_n = (\sin n) / n$

(o) $t_n = 1/n!$

(p) $t_n = 2^n / n!$

(q) $t_n = n! / 2^n$

(r) $t_n = (n^2 + 1)^{1/2} – n$

(s) $t_n = (n^2 + n + 1)^{1/2} – n$

(t) $t_n = (n^2 + 5n + 1)^{1/2} – (n^2 + n + 1)^{1/2}$

(u) $t_n = n \sin(1/n)$

(v) $t_n = \ln(n+1) - \ln n$

(w) $t_n = \ln(n^3+n+1) - \ln(n^2-n+5)$

(x) $t_n = n^{1/n}$

(y) $t_n = (1 + 3/n)^n$

(z) $b_n = n! / n^n$

4. For each of the following *recursively defined sequences*, determine convergence or divergence. In the former case, try to find the limit of the sequence.

(a) $t_1 = 1, t_2 = 1, t_n = t_{n-1} + t_{n-2}$ for all $n \geq 3$

(b) $a_1 = 4, a_n = a_{n-1} /2$ for all $n \geq 2$

(c) $c_1 = 3, c_n = 1.01(c_{n-1})$ for all $n \geq 2$

(d) $b_1 = 1, b_n = (b_{n-1} + 3/b_{n-1})/2$ for all $n \geq 2$

(e) $h_1 = 1, h_n = h_{n-1} + 1/n$ for all $n \geq 2$

(f) $a_1 = 1, a_2 = 2, a_n = (a_{n-1} + a_{n-2})/2$ for all $n \geq 3$.

(g) $z_1 = 1/3, z_n = (z_{n-1})^2$ for all $n \geq 2$.

5. Suppose that a sequence $\{x_n\}$ is defined recursively by:

$$x_0 = 1, x_1 = 2, \text{ and } x_{n+1} = 3x_n + x_{n-1}$$

Find $\lim x_{n+1}/x_n$ as $n \to \infty$.

Mathematicians have tried in vain to this day to discover some order in the sequence of prime numbers, and we have reason to believe that it is a mystery into which the human mind will never penetrate.

- Leonhard Euler (1707-1783)

At 6 P.M. the well marked 1/2 inch of water, at nightfall 3/4 and at daybreak 7/8 of an inch. By noon of the next day there was 15/16 and on the next night 31/32 of an inch of water in the hold. The situation was desperate. At this rate of increase few, if any, could tell where it would rise to in a few days.

- Stephen Leacock