MATHEMATICA LAB II

IMPROPER INTEGRALS

(Lab report due: March 16th)

(A) Evaluate exactly each of the following improper integrals.

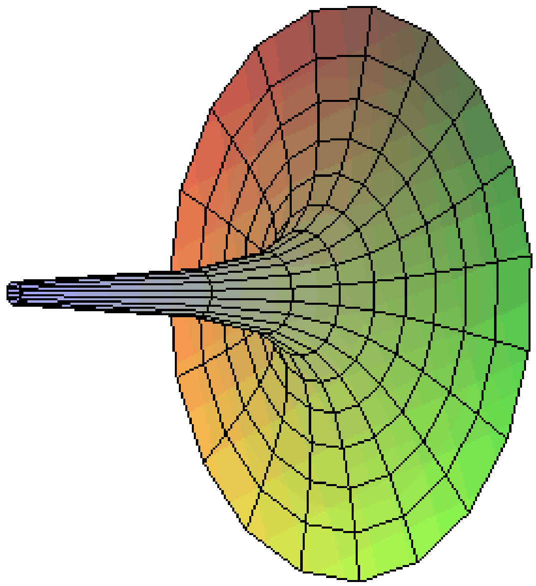
(1) $\displaystyle\int_0^{\infty} \frac{1}{x^2+x+1}\,dx$

(2) $\displaystyle\int_0^{\infty} \exp\left(-\frac{(x-1)^2}{5}\right)dx$ (Recall that exp(f(x)) means $e^{f(x)}$).

(3) $\displaystyle\int_{0+}^{1} \frac{1}{x+\sqrt{x}}\,dx$

(4) $\displaystyle\int_0^{\infty} \frac{1}{(x+1)(x+2)(x+3)}\,dx$

(B) Torricelli's Trumpet or Gabriel's Horn



Evangelista Torricelli (1608 - 1647), a student of Galileo, made a discovery that amazed him. Let's examine what occurred.

Rotate the curve $y = 1/x$ from $x = 1$ to infinity about the x-axis, thus obtaining an infinite solid of revolution. Let us call this solid the "horn of Gabriel", after the messenger of God in the old and new testaments.

Exercises: (5) Using disks or shells, compute the volume of this solid. Is this volume finite or infinite?

(6) Calculate the surface area of the horn of Gabriel. Is this surface area finite or infinite?

(7) Why was Torricelli amazed? Describe the paradox.

(C) The Gamma Function

Many important functions in applied mathematics and statistics are defined in terms of improper integrals. Perhaps the most famous of these is the *gamma function*, defined by:

$$\Gamma(x) = \int_0^\infty t^{x-1} \exp(-t)\, dt$$

This function is defined for x > 0.

Exercises: *You may use the built-in gamma function, Gamma[x], in Mathematica.*

(8) Plot the graph of y = Γ(x) over the interval [0.5, 6].

(9) By calculating Γ(1), Γ(2), Γ(3), Γ(4), and Γ(5), *guess* a formula for Γ(n) when *n* is a positive integer.

(10) Calculate Γ(1/2), Γ(3/2) and Γ(5/2).

(11) Consider Γ(x+1) / Γ(x). Substitute different positive values of x into this expression and observe what happens. Can you guess a general result for simplifying this expression? Using your conjecture, how is Γ(x+1) related to Γ(x) ?

(12) Stirling's formula states that

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

(Recall that two functions *f* and *g* are said to be **asymptotic** (and written **f ~ g**) if the limit of f(x) / g(x), as ∞ → 0, equals 1.)

Verify this statement by graphing the quotient of these two functions for appropriate values of *x*.

(D) Normal Distribution

A function p(x) is said to be a ***probability density function*** (or pdf) if:

- p(x) is defined for all real *x*
- $p(x) \geq 0$ for all real *x*
- the (improper) integral of *p* over the real line equals 1.

Exercises: Show that each of the following functions is a probability density function. Plot the graph of each pdf as well.

(13) $p_1(x) = (1/\text{sqrt}(2\pi)) \exp(-(x - 15)^2/2)$ defined for all *x*.

(14) $p_2(x) = (1/(3\text{sqrt}(2\pi)) \exp(-(x - 11)^2/18)$ defined for all *x*.

A pdf is often used to model a random event. For example, let us consider the following probabilistic model. Suppose that *R* is the amount of rainfall (in inches) in Alphaville during one year. (*R* is called a *random variable*.) Then the probability that $a \leq R \leq b$ is given by the *area* under the pdf, p_1, defined above from x = a to x = b.

Exercises: (15) Find the probability that the amount of rainfall in Alphaville is *more than* 16 inches during a year.

(16) Find the probability that the amount of rainfall in Alphaville is *less than* 13 inches during a year.

(17) The *mean value* (or *average value*) of a random variable with pdf p(x) is given by the (improper) integral of xp(x) over the real line. Find the mean value of the random variable R defined above.

(18) The *variance* of a random variable with pdf p(x) is given by the (improper) integral of $(x-\mu)^2 p(x)$ over the real line, where μ denotes the mean value of the random variable. Find the variance of R.

(19) Suppose that the amount of snowfall, *S*, (in inches) during one year in Betaville is given by the pdf $p_2(x)$ defined above. Find the *mean value* and the *variance* of *S*.