MATH 162 SOLUTIONS: QUIZ IV FRIDAY THE13TH

Answer any five of the following six questions. You may answer all six to obtain extra credit.

1. Explain *why the following improper integral diverges:*

$$\int_e^{\infty} \frac{1}{\sqrt{\ln x}}\, dx$$

Solution:

First note that $x > \ln x$ for all $x \geq e$. Hence:

$$\sqrt{x} > \sqrt{\ln x}$$

and so:

$$0 < \frac{1}{\sqrt{x}} \leq \frac{1}{\sqrt{\ln x}} \quad \text{for } x \geq e$$

Recalling that

$$\int_e^{\infty} \frac{1}{x^{1/2}}\, dx$$

diverges by the p-test, we now invoke the Comparison Test to obtain the desired result.

2. Compute the value of the following convergent improper integral. Assume that b is a positive constant.

$$\int_0^{\infty} e^{-bx} dx$$

Solution:

Using the definition of improper integral, we find:

$$\int_0^\infty e^{-bx}dx = \lim_{n\to\infty}\int_0^n e^{-bx}dx = \lim_{n\to\infty}\left(-\frac{1}{b}e^{-bx}\right)\Bigg|_0^n =$$

$$\lim_{n\to\infty}\left(-\frac{1}{b}\right)\left(e^{-bn}-e^{-0}\right) = \lim_{n\to\infty}\frac{1}{b}\left(1-\frac{1}{e^{bn}}\right) = \frac{1}{b}$$

The last limit uses the fact that b > 0.

3. Evaluate (i.e. find the exact value) the following convergent improper integral. Show your work! Calculator solutions are not acceptable.

$$\int_0^\infty \frac{x}{\left(x^2+13\right)^{3/2}}\,dx$$

Solution:

Using the definition of improper integral:

$$\int_0^\infty \frac{x}{\left(x^2+3\right)^{3/2}}\,dx = \lim_{n\to\infty}\int_0^n \frac{x}{\left(x^2+3\right)^{3/2}}\,dx =$$

$$\lim_{n\to\infty}\int_0^n x\left(x^2+3\right)^{-\frac{3}{2}}\,dx =$$

$$\lim_{n\to\infty}\left(-\left(x^2+3\right)^{-\frac{1}{2}}\right)\Bigg|_0^n = \lim_{n\to\infty}\left(-\left(n^2+3\right)^{-1/2}+3^{-1/2}\right) = \frac{1}{\sqrt{3}}$$

4. Evaluate (i.e. find the exact value) the following convergent improper integral. Show your work! Calculator solutions are not acceptable.

$$\int_0^\infty \frac{\arctan x}{1+x^2}dx$$

Solution:

Using the definition of improper integral:

$$\int_0^\infty \frac{\arctan x}{1+x^2}dx = \lim_{n\to\infty} \frac{1}{2}(\arctan x)^2\Big|_0^n =$$

$$\frac{1}{2}\lim_{n\to\infty}\left((\arctan n)^2 - 0\right) = \frac{1}{2}\left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{8}$$

5. Determine convergence or divergence. *Justify your answer! (That is, if you use the comparison test, exhibit the function that you choose to use for comparison and show why the appropriate inequality holds.)* Calculator solutions are not acceptable.

$$\int_{13}^\infty \frac{13+x+x^2}{(1313+x)^4}dx$$

Solution:

To apply the comparison test, observe that, for all $x \geq 13$:

$$0 \leq \frac{13+x+x^2}{(1313+x)^4} \leq \frac{13x^2+x^2+x^2}{x^4} = \frac{15x^2}{x^4} = 15\frac{1}{x^2}$$

Applying the p-test, the improper integral

$$\int_{13}^{\infty} \frac{1}{x^2}\, dx$$

converges, and hence, invoking the Comparison Test, the original improper integral must converge.

6. Determine convergence or divergence. *Justify your answer! (That is, if you use the comparison test, exhibit the function that you choose to use for comparison and show why the appropriate inequality holds.)* Calculator solutions are not acceptable.

$$\int_{13}^{\infty} \frac{13+x+e^x}{1313+x^5+13e^x}\, dx$$

Solution:

To apply the comparison test, observe that, for all $x \geq 13$:

$$\frac{13+x+e^x}{1313+x^5+13e^x} \geq \frac{e^x}{1313e^x+e^x+e^x} = \frac{1}{1315} > 0$$

Since, the improper integral

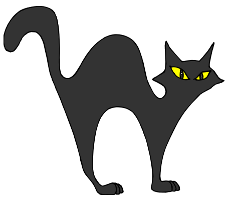
$$\int_{13}^{\infty} \frac{1}{1315}\, dx$$

clearly diverges, the original improper integral must diverge as well.

Extra Extra Credit:

$$\int_0^{1-} \frac{1}{\sqrt{1-x^4}}\ dx$$ *(Hint: Try using the Comparison Test.)*

Solution:



Since $x^2 > x^4$ for $0 \le x < 1$, $1 - x^2 < 1 - x^4$, and thus

$$0 < \frac{1}{\sqrt{1-x^4}} < \frac{1}{\sqrt{1-x^2}}$$ *for $0 \le x < 1$.*

Now $\displaystyle\int_0^{1-} \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{b \to 1-} \int_0^{b} \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{b \to 1-} (\arcsin b - \arcsin 0) = \frac{\pi}{2}$.

Thus, invoking the Comparison Test, the original integral converges also.