MATH 162 SOLUTIONS: QUIZ V

Part I: Select *any three* of the following four integrals. For each improper integral that you select, determine convergence or divergence. *Justify your answers!* (You may answer all four for extra credit.)

(a) $\displaystyle\int_0^{3-} \frac{1}{(3-x)^4}\,dx$

Solution: This integral diverges since:

$$\int_0^{3-} \frac{1}{(3-x)^4}\,dx = \lim_{c\to 3-}\int_0^{c} \frac{1}{(3-x)^4}\,dx = \lim_{c\to 3-} \left.\frac{1}{3(3-x)^3}\right|_0^c =$$

$$\lim_{c\to 3-}\left(\frac{1}{3(3-c)^3}-\frac{1}{3(3-0)^3}\right)\to\infty$$

(b) $\displaystyle\int_{0+}^{1} \frac{1+5x+7x^{15}}{\sqrt{x}}\,dx$

Solution: Note that the dominant term in the numerator is 1 (not x^{15}).

This integral converges because:

$$0<\frac{1+5x+7x^{15}}{\sqrt{x}}\le\frac{13}{\sqrt{x}} \quad \textit{for}\ \ 0<x\le 1$$

Using the p-test, we know that $\displaystyle\int_{0+}^{1}\frac{1}{\sqrt{x}}\,dx$ *converges. Hence, by the comparison test,*

$$\int_{0+}^{1} \frac{1+5x+7x^{15}}{\sqrt{x}}\,dx \quad \textit{converges}.$$

(c) $\displaystyle\int_{0}^{\frac{\pi}{4}-} \tan 2x\,dx$

Solution: This integral diverges because:

$$\int_{0}^{\frac{\pi}{4}-} \tan 2x\,dx = \lim_{c\to\frac{\pi}{4}-} \int_{0}^{c} \tan 2x\,dx = \lim_{c\to\frac{\pi}{4}-} \frac{1}{2}(-\ln|\cos 2x|)\Big|_{0}^{c} =$$

$$-\frac{1}{2}\lim_{c\to\frac{\pi}{4}-}\left(\ln|\cos 2c| - \ln(\cos 0)\right) = -\frac{1}{2}\lim_{c\to\frac{\pi}{4}-}\left(\ln|\cos 2c|\right) \to \infty$$

since $\cos(2c) \to 0+$ *as* $c \to (\pi/4)-$.

(d) $\displaystyle\int_{0+}^{\infty} \frac{1}{x^{1/4}+x+x^{2}}\,dx$

Solution: This integral of mixed type converges because, by definition:

$$\int_{0+}^{\infty} \frac{1}{x^{\frac{1}{4}}+x+x^{2}}\,dx = \int_{0+}^{1} \frac{1}{x^{\frac{1}{4}}+x+x^{2}}\,dx + \int_{1}^{\infty} \frac{1}{x^{\frac{1}{4}}+x+x^{2}}\,dx$$

and each of these two integrals converges (by virtue of the Comparison Test):

For $0 < x \le 1$:

$$0 < \frac{1}{x^{\frac{1}{4}}+x+x^{2}} < \frac{1}{x^{\frac{1}{4}}}$$

and by the p-test for integrals of type II,

$$\int_{0+}^{1} \frac{1}{x^{\frac{1}{4}}}\,dx \text{ converges.}$$

For $x \geq 1$:

$$0 < \frac{1}{x^{\frac{1}{4}} + x + x^2} < \frac{1}{x^2}$$

and by the p-test for integrals of type I,

$$\int_{1}^{\infty} \frac{1}{x^2}\,dx \text{ converges.}$$

Part II: For each of the following *sequences*, determine *convergence* or *divergence*. In the case of convergence, find the *limit* of the sequence. Briefly justify each answer. *(Select any 7 of the 8 sequences. For extra credit, you may solve all eight.)*

(a) $$a_n = \frac{n + 2015\ln n}{n!}$$

Solution: Since n = o(n!) and ln n = o(n), $\{a_n\}$ converges and its limit is 0.

(b) $$b_n = 3^{-n} + \ln\left(\frac{n+1789}{n+1492}\right) + \left(1 + \frac{4}{n}\right)^n$$

Solution: Using the fact that the limit of the sum of two convergent sequences is the sum of their limits, we have:

$$\lim b_n = \lim 3^{-n} + \lim \ln\left(\frac{n+1789}{n+1492}\right) + \lim\left(1 + \frac{4}{n}\right)^n = 0 + \ln 1 + e^4 = e^4$$

Thus the sequence $\{b_n\}$ converges and its limit is e^4.

(c) $$c_n = \frac{\sin 4n}{n}$$

Solution: Since $-1 \leq \sin(4n) \leq 1$, we have:

$$-\frac{1}{n} \leq \frac{\sin 4n}{n} \leq \frac{1}{n}$$

Applying the Squeeze Theorem, we conclude that $\{c_n\}$ converges to 0.

(d) $$d_n = \frac{n^n}{n^{14}+1}$$

Solution: Observing that $d_n \geq n^n/(n^{14} + n^{14}) = \frac{1}{2} n^n/n^{14} = \frac{1}{2} n^{n-14} \to \infty$ as $n \to \infty$, we conclude that d_n is unbounded, and thus divergent.

(e) $e_n = (-1)^n \cos(1/n)$

Solution: First note that, as $n \to \infty$, $\cos(1/n) \to \cos 0 = 1$.

Thus for large n, e_n is approximately $(-1)^n$ which is a divergent sequence.

(f) $$A_n = \frac{(n^2+7n+1)^3(3n+77)(2n^5+n+4)^2}{1+\ln n+5n^8(n+1)^9}$$

Solution: By selecting the dominant terms, we have:

$$\frac{(n^2+7n+1)^3(3n+77)(2n^5+n+4)^2}{1+\ln n+5n^8(n+1)^9} \cong \frac{(n^2)^3(3n)(2n^5)^2}{5n^8(n)^9} = \frac{12n^{17}}{5n^{17}} = \frac{12}{5}$$

Hence we conclude that $\{A_n\}$ converges to 12/5.

(g) $B_n =$ arctan (ln(n))

Solution: As n $\to \infty$, ln n $\to \infty$. and hence:

$$B_n = \arctan(\ln n) \to \frac{\pi}{2}$$

Thus the sequence $\{B_n\}$ converges and its limit is $\pi/2$.

(h) $C_n = \sqrt{n^2+9n+31} - \sqrt{n^2+3n+5}$

Solution: Rationalizing this expression:

$$h_n = \sqrt{n^2+9n+31} - \sqrt{n^2+3n+5} =$$

$$\left(\sqrt{n^2+9n+31} - \sqrt{n^2+3n+5}\right)\frac{\sqrt{n^2+9n+31} + \sqrt{n^2+3n+5}}{\sqrt{n^2+9n+31} + \sqrt{n^2+3n+5}} =$$

$$\frac{\left(n^2+9n+31\right)-\left(n^2+3n+5\right)}{\sqrt{n^2+9n+31} + \sqrt{n^2+3n+5}} = \frac{6n+26}{\sqrt{n^2+9n+31} + \sqrt{n^2+3n+5}}$$

$$\cong \frac{6n}{n+n} = 3$$

Thus the sequence $\{C_n\}$ converges and its limit is 3.

Mathematicians have tried in vain to this day to discover some order in the sequence of prime numbers, and we have reason to believe that it is a mystery into which the human mind will never penetrate.

- Leohard Euler (1707-1783)