MATH 162 SOLUTIONS: QUIZ VI

Instructions: For each of the following infinite series, determine *convergence* or *divergence*. You need not find the sum of those series that converge. *You must justify your answers!* Answer any 6 of the following 7 questions. You may answer all 7 to obtain extra credit. *[10 pts per problem]*

1. $$\sum_{n=1}^{\infty}\left(\frac{1}{n+1788}-\frac{1}{n+1789}\right)$$

Solution: This series is telescoping. Consider the sequence of partial sums:

$$s_1 = \frac{1}{1789}-\frac{1}{1790}$$

$$s_2 = \left(\frac{1}{1789}-\frac{1}{1790}\right)+\left(\frac{1}{1790}-\frac{1}{1791}\right)=\frac{1}{1789}-\frac{1}{1791}$$

$$s_3 = \left(\frac{1}{1789}-\frac{1}{1790}\right)+\left(\frac{1}{1790}-\frac{1}{1791}\right)+\left(\frac{1}{1791}-\frac{1}{1792}\right)=\frac{1}{1789}-\frac{1}{1792}$$

from which we infer that $s_n = 1/1789 - 1/(1789+n) \to 1/1789$ as $n \to \infty$. Thus our series converges and its sum is ***1/1789***.

2. $$\sum_{n=1}^{\infty}\frac{17^n n^2}{4^{2n}}$$

Solution:

$$\frac{17^n n^2}{4^{2n}}=\frac{17^n n^2}{16^n}=\left(\frac{17}{16}\right)^n n^2 \to \infty$$

Applying the n^th term test for divergence, we see that our series diverges.

Alternatively: Applying the ratio test

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{17^{n+1}(n+1)^2}{4^{2(n+1)}}}{\dfrac{17^n n^2}{4^{2n}}} = \frac{4^{2n}}{4^{2n+2}} \frac{17^{n+1}}{17^n} \frac{(n+1)^2}{n^2} =$$

$$\frac{17}{16}\left(\frac{n+1}{n}\right)^2 \rightarrow \frac{17}{16} > 1$$

we find that the series diverges (since r > 1).

3. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^n$

Solution: Since $0 < \left(\frac{1}{n}\right)^n = \frac{1}{n^n} \leq \frac{1}{n^2}$ *for all* $n \geq 1$

We already know (from an application of the Comparison Test discussed in class) that

$\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^2$ *converges.*

Finally, we can apply the Comparison Test to conclude that $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^n$ *converges.*

4. 19.113113113113…

Solution: All repeating decimals converge to a rational number.

Or we can just note that separating 19.1 from the decimal expansion yields a geometric series, viz.

19.113113113113… = 19 + {0.113 + 0.000113 + 0.000000113 + …}

Now 0.113 + 0.000113 + 0.000000113 + ... is geometric with ratio

$10^{-3} < 1$. Hence we conclude that the repeating decimal converges. If we wish to compute the sum of the series (which is not required in this problem) we have:

$$19.113113113... = 19 + \frac{113(10^{-3})}{1-10^{-3}} = 19 + \frac{113}{999}$$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{\sqrt{1+n+n^9}}$

Solution: $\displaystyle\frac{n^3}{\sqrt{1+n+n^9}} < \frac{n^3}{\sqrt{n^9}} = \frac{1}{n^{1.5}}$

Now the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.5}}$ *converges by the p-test. (Since we have not yet learned the p-test for $1 < p < 2$, everyone receives full credit for this problem.)*

6. $\displaystyle\sum_{n=1}^{\infty} \ln\frac{n}{n+1}$

Solution: Consider the partial sums:

$$s_1 = \ln 1 - \ln 2 = -\ln 2$$
$$s_2 = -\ln 2 + (\ln 2 - \ln 3) = -\ln 3$$
$$s_3 = -\ln 2 + (\ln 2 - \ln 3) + (\ln 3 - \ln 4) = -\ln 4$$
$$s_4 = -\ln 2 + (\ln 2 - \ln 3) + (\ln 3 - \ln 4) + (\ln 4 - \ln 5) = -\ln 5$$

From this we infer that $s_n = -\ln(n+1)$. Since the sequence $\{ s_n \}$ clearly diverges, we conclude that our original series diverges.

7. $$\sum_{n=1}^{\infty} \sqrt{\frac{n(n^2+1)^3}{(2n+2015)^7 + \ln x + (\ln \ln x)^{1789}}}$$

Solution:

$$\sqrt{\frac{n(n^2+1)^3}{(2n+2015)^7 + \ln n + (\ln \ln n)^{1789}}} > \sqrt{\frac{n(n^2)^3}{(2n+n)^7 + n + n}} =$$

$$\sqrt{\frac{n^7}{(3)^7 n^7 + n + n}} > \sqrt{\frac{n^7}{(3)^7 n^7 + n^7 + n^7}} > \sqrt{\frac{n^7}{(3)^8 n^7}} = \frac{1}{3^4} \neq 0$$

Applying the nth *term test for divergence, we see that our series diverges.*

EXTRA EXTRA CREDIT *(University of Michigan, Calculus 2 final exam question)*

Suppose that $f(x)$, $g(x)$, $h(x)$ and $k(x)$ are all positive, differentiable functions.

Suppose that

$$0 < f(x) < \frac{1}{x} < g(x) < \frac{1}{x^2}$$

for all $0 < x < 1$, and that

$$0 < h(x) < \frac{1}{x^2} < k(x) < \frac{1}{x}$$

for $x > 1$. Determine whether the following statements are always, sometimes or never true by circling the appropriate answer. No justification is necessary.

a. [2 points] $\int_0^1 g(x)dx$ converges.

Always **Sometimes** **[Never]**

b. [2 points] $\int_0^1 f(x)dx$ diverges.

Always **[Sometimes]** **Never**

c. [2 points] $\sum_{n=1}^{\infty} h(n)$ diverges.

Always **Sometimes** **[Never]**

d. [2 points] $\sum_{n=1}^{\infty} k(n)$ converges.

Always **[Sometimes]** **Never**

A mathematician is a blind man in a dark room looking for a black cat which isn't there.

- Charles R. Darwin