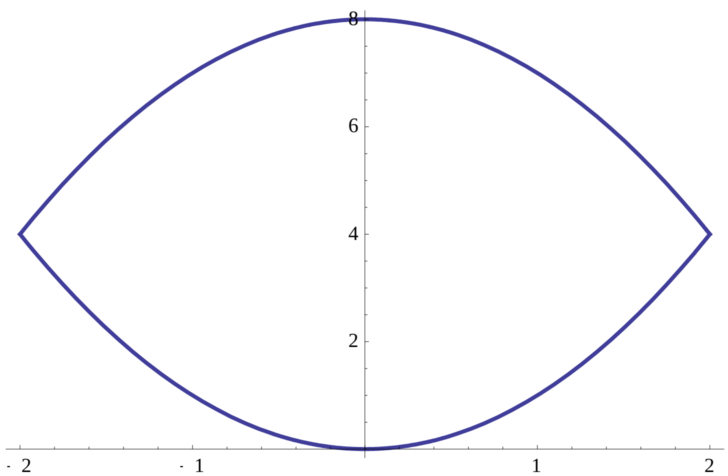
MATH 162 (PARTIAL) SOLUTIONS: TEST I

Instructions: Answer any 8 of the following 10 questions. You may answer more than 8 to obtain extra credit. You must show your reasoning; calculator answers are not acceptable.

1. Consider the region bounded by the curves $y = 8 - x^2$ and $y = x^2$. *Sketch!* Find the *volume* of the solid generated by revolving this region about the line x = 5. Express your answer as a Riemann integral. *(Do not evaluate.)*



2. Evaluate $\int x \cosh(3x)\, dx$

(Hint: recall that d/dx sinh x = cosh x and d/dx cosh x = sinh x)

Solution: Using integration by parts, we let f(x) = x and g′(x) = cosh 3x. Then f ′(x) = 1 and g(x) = (sinh 3x)/3. Thus

$$\int x \cosh(3x)\, dx = \frac{x\sinh(3x)}{3} - \int \frac{\sinh(3x)}{3}\, dx = \frac{x\sinh(3x)}{3} - \frac{\cosh(3x)}{9} + C$$

3. For each of the following statements answer *True* or *False*. *Briefly justify each answer!*

(a) $x^3 \ln x + x + 1 = o(x^4)$

True since $(x^3 \ln x + x + 1) / x^4 \to 0$ *as* $x \to \infty$.

(b) $\sinh x = O(\cosh x)$

True since $\sinh x / \cosh x = (e^x - e^{-x})/(e^x + e^{-x}) \to 1$ *as* $x \to \infty$.

(c) $$\frac{3x^3(x^2+1)^5 + 5x\ln x + 99}{x^5 + 5x^3 + x + 2015} = O(x^8)$$

True since:

$$\frac{\dfrac{3x^3(x^2+1)^5 + 5x\ln x + 99}{x^5 + 5x^3 + x + 2015}}{x^8} = \frac{3x^3(x^2+1)^5 + 5x\ln x + 99}{x^{13} + 5x^{11} + x^9 + 2015x^8} \to 3 \ \ as \ \ x \to \infty$$

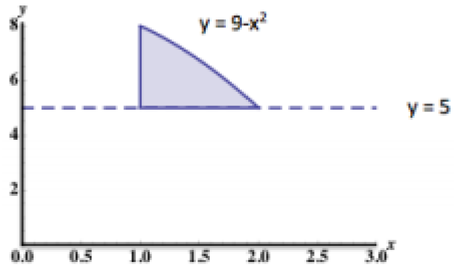
(d) $x = o((\ln x)^{2015})$

False since:

$$\frac{x}{(\ln x)^{2015}} \to 0 \ \ as \ \ x \to \infty$$

University of Michigan test problem:

4. [12 points] Consider the region in the xy-plane bounded by the curves $y = 9 - x^2$, $x = 1$, and $y = 5$. This region is pictured below.



Give a definite integral that computes the quantities below. You do not need to evaluate these integrals.

a. [3 points] The area of the region shown.

b. [3 points] The volume of the solid obtained by rotating the region about the y-axis.

c. [3 points] The volume of the solid obtained by rotating the region about the x-axis.

d. [3 points] The volume of the solid obtained by rotating the region about the line $y = 5$.

5. Evaluate $\int \arcsin x \; dx$

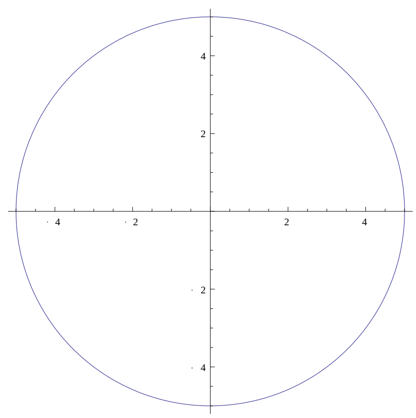
Solution:

Integration by parts: Let $f(x) = \arcsin x$ and $g'(x) = 1$. Then $f'(x) = 1/(1-x^2)^{1/2}$ and $g(x) = x$. So

$$\int \arcsin x \; dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx = x \arcsin x + \sqrt{1-x^2} + C$$

6. The base of a solid is a disk of radius 5. Each cross section cut by a plane perpendicular to a diameter is an isosceles right triangle with hypotenuse on the base. Express the volume of the solid as a Riemann integral. *(Do not evaluate.)*

Solution:



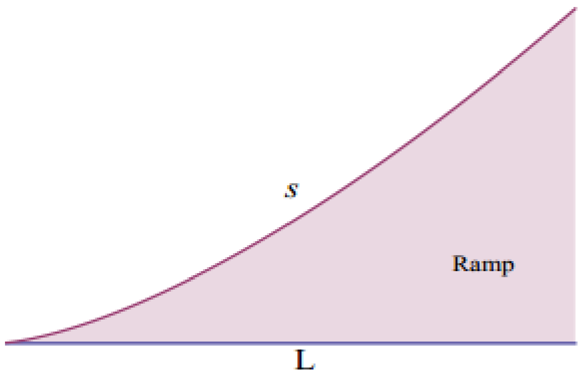
The equation of this circle is $x^2 + y^2 = 25$. Let us assume that the diameter referred to in the question lies on the x-axis. Then, taking a typical slice at x (in the interval [-5, 5], with thickness Δx, the volume of the corresponding slice (an isosceles right triangle with hypotenuse $2y = 2\,\mathrm{Sqrt}(25 - x^2)$ is given by

$\Delta V = \frac{1}{2}\, y\,(2y)\, \Delta x = (25 - x^2)\, \Delta x$. Thus:

$$V = \int_{-5}^{5} (25 - x^2)\, dx$$

7. *University of Michigan test problem:*

The RideJoyfully company wants to design a bicycle ramp using the shape of the function $f(x) = \frac{4}{3}x^{\frac{3}{2}}$ where L is the length in meters of the base of the ramp.
Find the length of the ramp.



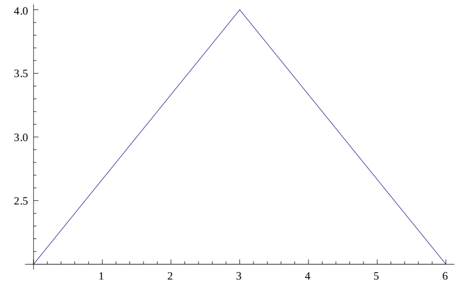
Solution:

$$ds = \sqrt{1+(dy/dx)^2}\ dx = \sqrt{1+(2\sqrt{x})^2}\ dx = \sqrt{1+4x}\ dx$$

$$s = \int_0^L \sqrt{1+4x}\ dx$$

8. Consider the triangle with vertices (0, 2), (6, 2), (3, 4). This triangle is rotated about the axis y = -3. Express the volume of this solid of revolution as a Riemann integral. *(Do not evaluate.)*

Solution:



The equations of the two non-horizontal sides are: y = (2/3)x + 2 and y = (-2/3)x + 6.

Solving for x, we obtain: x = (3/2) (y – 2) and x = -(3/2) (y – 6), respectively.

Using shells, the radius of the shell at y is y – (-3) = y + 3 and the length of the shell is -(3/2) (y – 6) – ((3/2) (y – 2)) = 12 – 3y. Hence:

$$V = \int_2^6 2\pi\left(y-(-3)\right)\left(12-3y\right)dy = 6\pi\int_2^6 \left(y+3\right)\left(4-y\right)dy$$

9. Assume that *m* and *n* are positive integers. Using integration by parts, derive the following reduction formula:

$$\int x^m (\ln x)^n dx = \frac{x^{m+1}(\ln x)^n}{m+1} - \frac{n}{m+1}\int x^m (\ln x)^{n-1} dx$$

Solution:

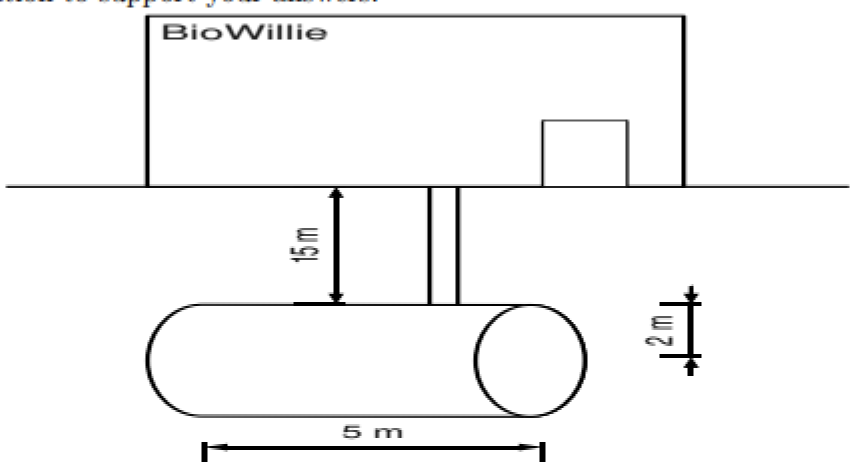
Let $f(x) = (\ln x)^n$ *and* $g'(x) = x^m$. *Thus* $f'(x) = n(\ln x)^{n-1}(1/x)$ *and* $g(x) = x^{m+1}/(m+1)$. *Thus:*

$$\int x^m (\ln x)^n dx = f(x)g(x) - \int f'(x)g(x)dx = (\ln x)^n \frac{x^{m+1}}{m+1} - \int \frac{n(\ln x)^{n-1}}{x}\frac{x^{m+1}}{m+1}dx =$$

$$\frac{x^{m+1}(\ln x)^n}{m+1} - \frac{n}{m+1}\int x^m (\ln x)^{n-1}dx$$

10. *University of Michigan test problem:*

[12 points] Country music legend Willie Nelson is concerned about our dependence of fossi fuels. In 2005, he started a company which sells a bio-diesel fuel called BioWillie. He recentl added a new cylindrical underground storage tank at his factory, and he needs to know how much work is required to pump all the fuel in a full tank to the surface. The tank is picture below. It is 5 meters long and has a radius of 2 meters. Its center line is 17 meters underground BioWillie fuel has a density of 900 kg per cubic meter. Make sure to include appropriate unit and justification to support your answers.



a. [7 points] Write an expression that approximates the work done in lifting a horizonta slice of fuel that is h_i meters below the ground's surface, given that the thickness of th slice is Δh meters.

Solution: Using the ground's surface as our horizontal axis and h be the variable on the vertical axis, then the cross section of the tank can be descirebed by
$x^2 + (h + 17)^2 = 4$.

$$So\ \Delta V = 2\sqrt{4-(h+17)^2}\ (5)\ \Delta h$$

Distance = -h

$$So\ \Delta W = (900)\ 2\sqrt{4-(h+17)^2}\ (5)(-h)\Delta h$$

b. [5 pts] Help Willie Nelson by computing the work required to pump all the fuel in a full tank to the ground's surface. Express your answer as a Riemann integral. *(Do not evaluate.)*

Solution:

$$So\ Work = -\int_{-19}^{-15}(9000)\ \sqrt{4-(h+17)^2}\ h\ dh$$

If a lion could talk, we could not understand him.

- **Ludwig Wittgenstein**