MATH 162 SOLUTIONS: TEST II

FRIDAY THE 13 MARCH 2015

PART I *(Answer all four problems.)*

1. Compute the value of the following improper integral:

$$\int_{0+}^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}}\, dx$$

Solution:

$$\int_{0+}^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}}\, dx = \lim_{c\to 0+} \int_{c}^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}}\, dx = \lim_{c\to 0+} 2\sqrt{\sin x}\Big|_{c}^{\pi/2} =$$

$$2\lim_{c\to 0+}\left(\sqrt{\sin\frac{\pi}{2}} - \sqrt{\sin c}\right) = 2$$

2. Albertine ponders the following recursively defined sequence:

$c_1 = 1,$

$$c_{n+1} = \frac{1}{c_n} + 3 \quad \text{for } n \geq 1$$

(a) Find the values of c_2, c_3 and c_4.

Solution:

Setting n = 1:

$$c_2 = \frac{1}{c_1} + 3 = 1 + 3 = 4$$

Setting n = 2:

$$c_3 = \frac{1}{c_2} + 3 = \frac{1}{4} + 3 = \frac{13}{4} \cong 3.25$$

Setting n = 3:

$$c_4 = \frac{1}{c_3} + 3 = \frac{4}{13} + 3 = \frac{43}{13} \cong 3.307$$

(b) Assuming that the limit of c_n as $n \to \infty$ exists, help Albertine to find its *exact* value.

Solution:

Assume that $L = \lim c_n$ exists. Then:

$$\lim c_{n+1} = \lim\left(\frac{1}{c_n} + 3\right)$$

and so:

$$L = \frac{1}{L} + 3$$

Multiplying both sides by L yields: $L^2 = 1 + 3L$. *So:* $L^2 - 3L - 1 = 0$. *Using the quadratic formula:*

$$L = \frac{3 \pm \sqrt{13}}{2} \cong 3.3027,\ -0.3027$$

We reject the negative root, since $c_1 > 0$ *and all subsequent terms of the sequence are also positive (reasoning inductively).*

Thus, if $\lim c_n$ *exists, this limit must be*

$$\frac{3+\sqrt{13}}{2}$$

3. Determine *convergence* or *divergence* of the following improper integral. Justify your answer:

$$\int_{1+}^{2} \frac{x^2}{(x^3-1)^2}\,dx \qquad \text{(\textit{Hint:} Integrate.)}$$

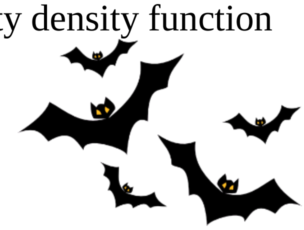
Solution:

$$\int_{1+}^{2} \frac{x^2}{(x^3-1)^2}\,dx = \lim_{c\to 1+} \int_{c}^{2} \frac{x^2}{(x^3-1)^2}\,dx = \lim_{c\to 1+} -\frac{(x^3-1)^{-1}}{3}\Bigg|_c^2 =$$

$$\lim_{c\to 1+} -\frac{1}{3}\left((2^3-1)^{-1} - (c-1)^{-1}\right) \ \textit{does not exist}$$

4. The life-span (in years) of a vampire bat can be modeled by a random variable X with probability density function

$$f(x) = \begin{cases} ce^{-x/10} & \textit{if } x \ge 0 \\ 0 & \textit{otherwise} \end{cases}$$



(a) Find the constant *c*. (*Hint:* Every bat must die.)

Solution:

$$1 = \int_0^\infty ce^{-x/10}\,dx = c\lim_{c\to\infty}\int_0^c e^{-x/10}\,dx = c\lim_{c\to\infty}\left(-10e^{-x/10}\Big|_0^c\right) =$$

$$c\lim_{c\to\infty} -10\left(e^{-c/10} - e^0\right) = 10c$$

Hence c = 1/10

(b) Find the probability that a randomly chosen vampire bat will *live longer than 11 years*. (Express your answer to the *nearest hundredth*.)

Solution:

$$P(bat\ lives\ longer\ than\ 11\ years) = \frac{1}{10}\int_{11}^\infty e^{-x/10}\,dx =$$

$$\frac{1}{10}\lim_{c\to\infty}\int_{11}^c e^{-x/10}\,dx = \frac{1}{10}\lim_{c\to\infty}\left((-10)e^{-x/10}\Big|_{11}^c\right) =$$

$$(-1)\lim_{c\to\infty}\left(e^{-c/10} - e^{-11/10}\right) = e^{-11/10} \approx 0.33$$

PART II *Select any 4 of the following 5 sequences.* For each selected sequence, determine *convergence* or *divergence*. *Briefly* justify each answer. *In the case of convergence, find the limit.* Calculator results will not earn full credit. (You may answer all 6 to earn extra credit.)

1. $a_n = n\sin\left(\dfrac{\pi}{13n}\right)$

Solution: Let h = 1/(13n). Then n = 1/(13h) and as n $\to \infty$, h $\to 0$. Hence:

$$a_n = n\sin\left(\frac{1}{13n}\right) = \frac{1}{13h}\sin h = \frac{1}{13}\left(\frac{\sin\, h}{h}\right) \to \frac{1}{13}\ as\ h \to 0$$

Hence the sequence a_n converges to 1/13.

2. $$b_n = \left(1 + \frac{\pi}{n}\right)^{2n}$$

Solution: Note that:

$$b_n = \left(1 + \frac{\pi}{n}\right)^{2n} = \left(\left(1 + \frac{\pi}{n}\right)^{n}\right)^{2} \to \left(e^{\pi}\right)^2 = e^{2\pi}$$

Hence the sequence b_n converges to $e^{2\pi}$.

3. $$c_n = \sqrt{n^4 + 19n^2 + \pi} - \sqrt{n^4 - 7n^2 + n + 13\pi}$$

Solution: Rationalizing the "numerator" yields:

$$c_n = \left(\sqrt{n^4 + 19n^2 + \pi} - \sqrt{n^4 - 7n^2 + n + 13}\right)\left(\frac{\sqrt{n^4 + 19n^2 + \pi} + \sqrt{n^4 - 7n^2 + n + 13}}{\sqrt{n^4 + 19n^2 + \pi} + \sqrt{n^4 - 7n^2 + n + 13}}\right) =$$

$$\frac{26n^2 - n + \pi - 13}{\sqrt{n^4 + 19n^2 + \pi} + \sqrt{n^4 - 7n^2 + n + 13}} \cong \frac{26n^2}{\sqrt{n^4} + \sqrt{n^4}} = 13$$

Hence the sequence c_n converges to 13.

4. $d_n = \ln(\pi n + 2015\pi) - \ln(5n - \pi) + \arctan\, n$

Solution: The sequence converges:

$$d_n = \ln(\pi n + 2015\pi) - \ln(5n - \pi) + \arctan n =$$

$$\ln \frac{\pi n + 2015\pi}{5n - \pi} + \arctan n \to \ln \frac{\pi}{5} + \frac{\pi}{2}$$

5. $f_n = \sqrt{13 + 44\frac{\ln n}{n} + \pi \cos\left(\frac{\pi}{n}\right) + \frac{\sin^4 \pi n}{n}}$

Solution: *The sequence converges:*

$\ln n/n \to 0$

$\cos(\pi/n) \to \cos(0) = 1$

$$\frac{\sin^4 \pi n}{n} \to 0$$

Thus

$$f_n = \sqrt{13 + 44\frac{\ln n}{n} + \pi \cos\left(\frac{\pi}{n}\right) + \frac{\sin^4 \pi n}{n}} \to \sqrt{13 + 0 + \pi + 0}$$

$$= \sqrt{13 + \pi}$$

PART III Select any 4 of the following 5 series. For each selected series, determine *convergence* or *divergence*. Justify each answer. (You may answer all 5 to earn extra credit.)

1. $\sum_{n=1}^{\infty} \left(\arctan(n-1) - \arctan(n)\right)$

Solution: Since this series is telescoping, we will consider the sequence of partial sums:

$s_1 = \arctan(0) - \arctan(1)$

$s_2 = (\arctan(0) - \arctan(1)) + (\arctan(1) - \arctan(2) = -\arctan(2)$

$s_3 = (\arctan(0) - \arctan(1)) + (\arctan(1) - \arctan(2)) + (\arctan(2) - \arctan(3))$

$= -\arctan(3)$

We infer that, in general, $s_n = -\arctan(n)$.

Now $s_n = -\arctan n \to -\pi/2$ *as* $n \to \infty$. *So the series is convergent.*

2. $\sum_{n=1}^{\infty} \frac{7\pi n^{\pi}}{4^{2n}}$

Solution: Applying the ratio test

$$\frac{a_{n+1}}{a_n} = \frac{\frac{7(n+1)^{\pi}}{4^{2(n+1)}}}{\frac{7n^{\pi}}{4^{2n}}} = \frac{4^{2n}}{4^{2n+2}} \frac{7}{7} \frac{(n+1)^{\pi}}{n^{\pi}} =$$

$$\frac{1}{16}\left(\frac{n+1}{n}\right)^{\pi} \to \frac{1}{16} < 1$$

we find that the series converges since $r < 1$.

3. $\sum_{n=1}^{\infty} \left(\frac{n+\pi}{n}\right)^n$

Solution:

Since $\ln x < x$ *, we have:*

$$0 < \frac{\pi + \ln x}{x^3} < \frac{\pi \ln x}{x^3} < \frac{\pi x}{x^3} = \pi \frac{1}{x^2}$$

Thus, invoking both the p-test and the Comparison Test, our original integral converges.

4. 314.314314314…

Solution: This is the geometric series: $314 + (314)10^{-3} + (314)10^{-6} + \ldots$
Since $r = 10^{-3} < 1$*, our series converges.*
Its sum is

$$\frac{314}{1-0.001} = \frac{314}{0.999} = \frac{314000}{999}$$

5. $\sum_{n=2}^{\infty} \frac{1+\sin^4(n)}{n\ln(n^{13})}$

Solution: Consider the following inequality:

$$\frac{1+\sin^4(n)}{n\ln(n^{13})} \geq \frac{1}{13n\ln n} > 0$$

Using the p-test, we see that the smaller series diverges and hence our series diverges as well.

PART IV. Select any three of the following four problems. You may answer all four for extra credit. For each improper integral below, determine convergence or divergence. ***Justify each answer!***

$$(A) \quad \int_1^\infty \frac{2015+\ln x}{x^3}\,dx$$

Solution: Since ln x < x for x > 1

$$0 < \frac{2015+\ln x}{x^3} < \frac{2015x+x}{x^3} = 2016\frac{1}{x^3}$$

Now using the Comparison Test, and the p-test for p = 3, we see that our improper integral converges.

$$(B) \quad \int_0^\infty \frac{1+x^2e^{\pi x}+(\ln x)^{2015\pi}}{\pi+e^{4x}}\,dx$$

Solution: Observe that

$$0 < \frac{1+x^2e^{\pi x}+(\ln x)^{2015\pi}}{\pi+e^{4x}} < \frac{e^{\pi x}+x^2e^{\pi x}+e^{\pi x}}{e^{4x}} =$$

$$\frac{1+x^2+1}{e^{(4-\pi)x}} < \frac{3x^2}{e^{(4-\pi)x}} < \frac{3e^{\frac{(4-\pi)}{2}x}}{e^{(4-\pi)x}} = 3\frac{1}{e^{\frac{(4-\pi)}{2}x}} < 3\frac{1}{e^{\frac{(1/2)}{2}x}} = 3\frac{1}{e^{x/4}}$$

$$0 < \frac{1+xe^{2x}}{1+e^{4x}} < \frac{xe^{2x}+xe^{2x}}{e^{4x}} = \frac{2xe^{2x}}{e^{4x}} = \frac{2x}{e^{2x}} < \frac{2e^x}{e^{2x}} = \frac{2}{e^x}$$

Now $\int_0^\infty e^{-x}\,dx$ *converges.*

Thus, invoking the Comparison Test, our original integral converges.

$$(C)\quad \int_0^\infty \frac{9+\pi x^6+13\sqrt{x}}{5\pi+4\sqrt{x}+x^8}\,dx$$

Solution: Observe that

$$0<\frac{9+\pi x^6+13\sqrt{x}}{5\pi+4\sqrt{x}+x^8}<\frac{9x^6+\pi x^6+13x^6}{x^8}=(22+\pi)\frac{1}{x^2}$$

Thus, invoking the Comparison Test, our original integral converges.

$$(D)\quad \int_2^\infty \frac{4\pi+3\cos^4(4x+1)}{x+\pi^\pi}\,dx$$

Solution:

Observe that

$$\frac{4\pi+3\cos^4(4x+1)}{x+\pi^\pi}>\frac{1}{x+\pi^\pi x}=\frac{1}{1+\pi^\pi}\frac{1}{x}>0$$

Thus, invoking the Comparison Test, our original integral diverges.

PART V. Select any 3 of the following 4 problems. You may answer all four for extra credit. For each numerical series below, determine *convergence* or *divergence*. Justify each answer.

(A) $$\sum_{n=1}^{\infty} \frac{n^5 2^n (n!)^2}{(2n)!}$$

Solution: Applying the ratio test to this positive series:

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{(n+1)^5 2^{n+1}((n+1)!)^2}{(2n+2)!}}{\dfrac{n^5 2^n (n!)^2}{(2n)!}} = \frac{(n+1)^5}{n^5} \frac{2^{n+1}}{2^n} \frac{((n+1)!)^2}{(n!)^2} \frac{(2n)!}{(2n+2)!} =$$

$$\left(\frac{n+1}{n}\right)^5 (2) \left(\frac{(n+1)!}{n!}\right)^2 \frac{1}{(2n+2)(2n+1)} = 2\left(\frac{n+1}{n}\right)^5 \frac{(n+1)^2}{2(n+1)(2n+1)} =$$

$$\left(\frac{n+1}{n}\right)^5 \frac{(n+1)^2}{(n+1)(2n+1)} \to 1^5 \frac{1}{2} = \frac{1}{2} = r$$

Since r < 1, we conclude that our positive series converges.

(B) $$\sum_{n=1}^{\infty} \frac{1}{\left(1+\dfrac{\pi}{n}\right)^{n^2}}$$

Solution: Applying the nth root test to this positive series:

$$(a_n)^{1/n} = \frac{1}{\left(1+\dfrac{\pi}{n}\right)^n} \to \frac{1}{e^{\pi}} = \rho < 1$$

Since ρ < 1, we conclude that our positive series converges.

(C) $\sum_{n=1}^{\infty} \frac{n^5}{e^{n^2}}$

Solution:

Applying the ratio test to this positive series:

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^5}{e^{(n+1)^2}}}{\frac{n^5}{e^{n^2}}} = \left(\frac{n+1}{n}\right)^5 \frac{e^{n^2}}{e^{(n+1)^2}} =$$

$$\left(\frac{n+1}{n}\right)^5 \frac{1}{e^{2n+1}} \to \left(1^5\right)(0) = 0$$

(D) $\sum_{n=1}^{\infty} \left(\frac{n+3}{n+\pi}\right)^{\pi}$

Solution: Since

$$\left(\frac{n+3}{n+\pi}\right)^{33} = \left(\frac{1+3/n}{1+\pi/n}\right)^{33} \to 1^{33} = 1,$$

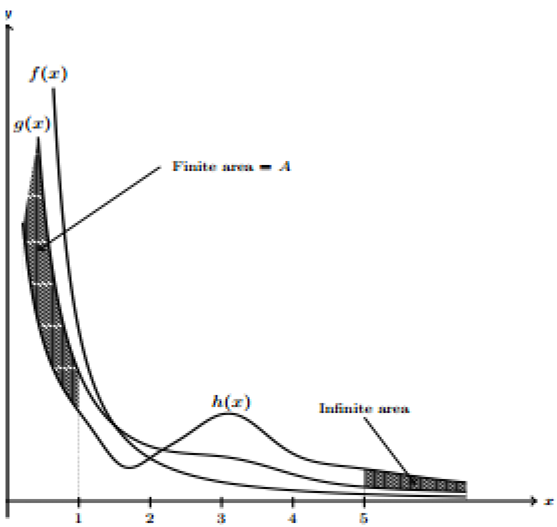
We may invoke the n th Term Test for Divergence to conclude that our original series diverges.

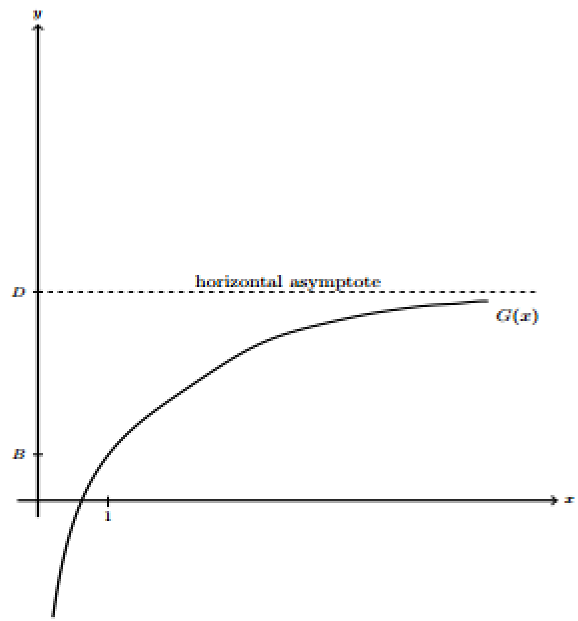
Since $\left(\frac{n+3}{n+\pi}\right)^{\pi} \to (1)^{\pi} \to 1$ *we apply the* n^{th} *Term Test for Divergence to conclude that our series diverges.*

EXTRA CREDIT (University of Michigan midterm problem)

[15 points] Graphs of f, g and h are below. Each function is positive, is continuous on $(0, \infty)$, has a horizontal asymptote at $y = 0$ and has a vertical asymptote at $x = 0$. The area between $g(x)$ and $h(x)$ on the interval $(0, 1]$ is a finite number A, and the area between $g(x)$ and $h(x)$ on the interval $[5, \infty)$ is infinite. On the right is a graph of an antiderivative $G(x)$ of $g(x)$. It also has a vertical asymptote at $x = 0$.

Use the information in these graphs to determine whether the following three improper integrals **converge**, **diverge**, or whether there is **insufficient information to tell**. You may assume that f, g and h have no intersection points other than those shown in the graph. **Justify all your answers.**





a. [3 points] $\displaystyle\int_1^\infty h(x)dx$

Solution: Diverges

$$\int_1^\infty h(x)dx = \int_1^5 h(x)dx + \int_5^\infty h(x)dx = \text{finite integral} + \text{divergent integral}$$

b. [4 points] $\displaystyle\int_0^1 g(x)dx$

Solution: Diverges

$$\int_0^1 g(x)dx = \lim_{b\to 0^+} \int_b^1 g(x)dx = \lim_{b\to 0^+} G(x)|_b^1 = \lim_{b\to 0^+} G(1) - G(b) = \infty \text{ Diverges}$$

c. [3 points] $\int_0^1 h(x)dx$

Solution: Diverges since

$$\int_0^1 h(x)dx = \int_0^1 g(x)dx - \int_0^1 (g(x) - h(x))dx = \text{ divergent integral+ finite integral}$$

d. [5 points] If $f(x) = 1/x^p$, what are all the possible values of p? **Justify your answer.**

Solution:

$$\begin{aligned}\int_1^\infty g(x)dx &= \lim_{b\to\infty} \int_1^b g(x)dx \\ &= \lim_{b\to\infty} G(b) - G(1) = D - B \text{ converges} \\ \int_3^\infty f(x)dx &< \int_3^\infty g(x)dx \text{ convergent integral}\end{aligned}$$

Hence $p > 1$.

I tell them that if they will occupy themselves with the study of mathematics they will find in it the best remedy against the lusts of the flesh.

Thomas Mann, **THE MAGIC MOUNTAIN**