MATH 162 SOLUTIONS: TEST III

"Then you should say what you mean," the March Hare went on.
"I do, " Alice hastily replied; "at least I mean what I say, that's the same thing, you know."

"Not the same thing a bit!" said the Hatter. "Why, you might just as well say that "I see what I eat" is the same thing as "I eat what I see!"

- Lewis Carroll, **Alice in Wonderland**

Instructions: *Answer any 7 of the following 8 problems. You may answer all 8 to obtain extra credit.*

1. Without using l'Hôpital's rule, find:

$$\lim_{x\to 0} \frac{e^{3x^2} - 1 - 3x^2 - \frac{9}{2}x^4 - \frac{9}{2}x^6}{\cos(x^4) - 1}$$

Solution:

Since

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + O(x^5)$$

it follows that:

$$e^{3x^2} = 1 + \frac{3x^2}{1!} + \frac{\left(3x^2\right)^2}{2!} + \frac{\left(3x^2\right)^3}{3!} + \frac{\left(3x^2\right)^4}{4!} + O(x^{10}) =$$

$$1 + 3x^2 + \frac{9x^4}{2} + \frac{27x^6}{6} + \frac{81x^8}{24} + O(x^{10}) =$$

$$1 + 3x^2 + \frac{9x^4}{2} + \frac{9x^6}{2} + \frac{27x^8}{8} + O(x^{10})$$

Since

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + O(x^6)$$

it follows that:

$$\cos\left(x^4\right) = 1 - \frac{x^8}{2!} + O(x^{16})$$

Hence:

$$\frac{e^{3x^2}-1-3x^2-\frac{9}{2}x^4-\frac{9}{2}x^6}{\cos(x^4)-1}=$$

$$\frac{1+3x^2+\frac{9x^4}{2}+\frac{9x^6}{2}+\frac{27x^8}{8}+O(x^{10})-1-3x^2-\frac{9}{2}x^4-\frac{9}{2}x^6}{1-\frac{x^8}{2!}+O(x^{16})-1}=$$

$$\frac{\frac{27x^8}{8}+O(x^{10})}{-\frac{x^8}{2!}+O(x^{16})}\to\frac{\frac{27}{8}}{-\frac{1}{2!}}=-\frac{27}{4}$$

2. Given y = G(x) below, calculate the value of $G^{(1313)}(0)$. *(Express your answer in factorial form.)*

$$G(x)=x^3\sinh(x^2)$$

Solution:

Beginning with the Maclaurin series for sinh t and then replacing t by x^2:

$$\sinh t = \frac{t}{1!} + \frac{t^3}{3!} + \frac{t^5}{5!} - \ldots + \frac{t^{2n+1}}{(2n+1)!} + \ldots$$

$$\sinh\left(x^2\right) = \frac{x^2}{1!} + \frac{x^6}{3!} + \frac{x^{10}}{5!} + \ldots + \frac{x^{4n+2}}{(2n+1)!} + \ldots$$

Now, multiplying by x^3 yields:

$$G(x) = x^3 \sinh\left(x^2\right) = \frac{x^5}{1!} + \frac{x^9}{3!} + \frac{x^{13}}{5!} + \ldots + \frac{x^{4n+5}}{(2n+1)!} + \ldots$$

Now, the general Maclaurin series of G(x) is:

$$G(x) = G(0) + \frac{G'(0)}{1!} x + \ldots + \frac{G^{(k)}(0)}{k!} x^k + \ldots$$

Thus the coefficient of x^{1313} is $G^{(1313)}(0) / 1313!$

Now the series for x^3 sinh (x^2) has coefficient of x^{1313} occur when 4n + 5=1313, that is, when n =327 (and so 2n + 1=655). Thus this coefficient is: 1 / 655!

Equating $G^{(1313)}(0) / 1313!$ with 1 /655!, we find that:

$G^{(1313)}(0) = 1313! \,/\, 655!$

3. By dividing power series, find the *first three non-zero* terms of the Maclaurin series of

$$\frac{e^{x^2}}{1+\sin x}$$

Solution:

$$\frac{e^{x^2}}{1+\sin x} = \frac{1+x^2+\frac{1}{2!}x^4+\frac{1}{3!}x^6+\ldots}{1+(x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\ldots)} =$$

$$\frac{1+x^2+\frac{1}{2}x^4+\frac{1}{6}x^6+\ldots}{1+x-\frac{1}{6}x^3+\ldots} = 1+x+x^2+\frac{5}{6}x^3+\ldots$$

4. For each series below, determine *absolute convergence*, *conditional convergence* or *divergence*. Justify each answer.

$$(a) \quad \sum_{n=3}^{\infty} (-1)^n \frac{13}{n\sqrt{\ln n}}$$

Solution:

Notice that this series fails to converge absolutely, by the p-test. It does converge, however, due to the Cauchy-Leibniz test. Thus the series converges conditionally.

$$(b) \quad \sum_{k=1}^{\infty} (-1)^k \arctan(k^2)$$

Solution:

Since arctan(k^2) $\rightarrow \pi/2$ as k $\rightarrow \infty$, the series diverges by the n^{th}-term test for divergence.

$$(c) \quad \sum_{n=1}^{\infty} (-1)^n \frac{(n!)^2}{(2n)!}$$

Solution:

Applying the Ratio Test, we see that the series converges absolutely:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\dfrac{\left((n+1)!\right)^2}{(2(n+1))!}}{\dfrac{(n!)^2}{(2n)!}} = \frac{(2n)!}{(2n+2)!} \frac{\left((n+1)!\right)^2}{\left(n!\right)^2} =$$

$$\frac{1}{(2n+2)(2n+1)} \left(\frac{(n+1)!}{n!}\right)^2 = \frac{(n+1)^2}{(2n+2)(2n+1)} \rightarrow \frac{1}{4} < 1$$

5. Write each of the following in the form a + bi. *Show your work!*

(a) 3(9 – 4i) – 5(-6 – 3i)

Answer: 57 + 3i

(b) (1 – i)(2 – 5i)

Answer: *-4 – 7i*

(c) $(3 - i)^3$

Answer: *18 – 26 i*

$(d)\quad \overline{14 - 345i}$

Answer: *14 + 345i*

$(f)\quad \dfrac{3-5i}{1+2i}$

Answer: $\dfrac{3-5i}{1+2i} = (-48/5) + (96/5)i$

(g) $i^{1789} + i^{444} - i^{9902}$

Answer: $i^{1789} + i^{444} - i^{9902} = i^{4(447)+1} + i^{4(111)} - i^{4(247)+2} = i + 1 + 1$
$= 2 + i$

6. For each power series below, determine the *interval of convergence*. *Do not* investigate the behavior of at endpoints.

$$(a)\quad \sum_{n=1}^{\infty} \frac{n^{13}}{13^n}(x-13)^n$$

Solution:

Using the ratio test:

$$\left|\frac{\frac{(n+1)^{13}}{13^{n+1}}(x-13)^{n+1}}{\frac{n^{13}}{13^n}(x-13)^n}\right| = \frac{1}{13}\frac{(n+1)^{13}}{n^{13}}|x-13| =$$

$$\frac{1}{13}(1+1/n)^{13}|x-13| \to \frac{1}{13}|x-13|$$

Thus the series converges absolutely for |x – 13| < 13. So the interval of convergence is $(0, 26)$.

(b) $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}(x-4)^n$

Solution:

Invoking the n^th root test:

$$\left|\left(1+\frac{1}{n}\right)^{n^2}(x-4)^n\right|^{1/n} = \left(1+\frac{1}{n}\right)^n |x-4| \to e|x-4|$$

Thus the series converges absolutely for e |x – 4| < 1. So the interval of convergence is (4 – 1/e, 4 + 1/e).

7. For each power series below, determine the *interval of convergence*. Investigate *end point behavior*.

$$(a)\quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+13}}(x-13)^n$$

Solution:

Using the ratio test:

$$\left|\frac{\frac{1}{\sqrt{n+14}}(x-13)^{n+1}}{\frac{1}{\sqrt{n+13}}(x-13)^n}\right| = \frac{\sqrt{n+13}}{\sqrt{n+14}}|x-13| = \sqrt{\frac{n+13}{n+14}}|x-13| \to |x-13|$$

Thus the series converges absolutely for |x – 13| < 1. So the interval of convergence is (12, 14).

At x = 14, the series equals:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+13}}(14-13)^n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+13}}$$

which diverges (using the comparison test and the p-test).

At x = 12, the series equals:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+13}}(12-13)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+13}}$$

which converges conditionally (using the Cauchy-Leibniz test as well as the fact that the series fails to converge absolutely).

(b) $\sum_{n=1}^{\infty} \frac{13^n}{n^2} x^{2n}$

Solution:

Using the ratio test:

$$\left| \frac{\frac{13^{n+1}}{(n+1)^2} x^{2(n+1)}}{\frac{13^n}{n^2} x^{2n}} \right| = 13 \frac{n^2}{(n+1)^2} x^2 = 13 \left(\frac{n}{n+1} \right)^2 x^2 \to 13x^2$$

Thus the series converges absolutely for $13x^2 < 1$. So the interval of convergence is $\left(-1/\sqrt{13},\ 1/\sqrt{13}\right)$.

At $x = 1/\sqrt{13}$,

$$\sum_{n=1}^{\infty} \frac{13^n}{n^2} x^{2n} = \sum_{n=1}^{\infty} \frac{13^n}{n^2} \left(\frac{1}{13} \right)^n = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

which converges absolutely (using the p-test).

At $x = -1/\sqrt{13}$,

$$\sum_{n=1}^{\infty} \frac{13^n}{n^2} x^{2n} = \sum_{n=1}^{\infty} \frac{13^n}{n^2} \left(\frac{-1}{\sqrt{13}} \right)^{2n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

which converges absolutely (using the p-test).

8. Find the *interval of convergence* of each of the following power series:

$$(a) \quad \sum_{n=1}^{\infty} \frac{n!}{(1)(3)(5)(7)...(2n-1)} x^{3n}$$

Solution: Using the ratio test:

$$\frac{\dfrac{(n+1)!}{(1)(3)(5)(7)...(2n-1)(2n+1)} |x|^{3n+3}}{\dfrac{n!}{(1)(3)(5)(7)...(2n-1)} |x|^{3n}} = (n+1)\frac{1}{2n+1}|x|^3 \to \frac{1}{2}|x|^3$$

Now, the series converges absolutely when $\frac{1}{2}|x|^3 < 1$.

Thus the interval of convergence of our series is:

$$\left(-\sqrt[3]{2}, \sqrt[3]{2}\right)$$

$$(b) \quad \sum_{n=1}^{\infty} \frac{7^n \sqrt{n^2+4}}{\left(n^{4/3}+1789\right)^3} (x-15)^n$$

Solution: Applying the ratio test,

$$\frac{\dfrac{7^{n+1}\sqrt{(n+1)^2+4}}{\left((n+1)^{4/3}+1789\right)^3}}{\dfrac{7^n\sqrt{n^2+4}}{\left(n^{4/3}+1789\right)^3}}\frac{|x-15|^{n+1}}{|x-15|^n}=7\left(\frac{\left(n^{4/3}+1789\right)^3}{\left((n+1)^{4/3}+1789\right)^3}\right)|x-15|$$

$$=7\left(\frac{n^{4/3}+1789}{(n+1)^{4/3}+1789}\right)^3|x-15|\to 7|x-15|$$

Thus the interval of convergence of our series is

$$\left(15-\frac{1}{7},\,15+\frac{1}{7}\right)$$

Extra Credit: Using a series representation of sin(3x), find constants *r* and *s* for which:

$$\lim_{x\to 0}\left(\frac{\sin(3x)}{x^3}+\frac{r}{x^2}+s\right)=0$$

Solution:

Since

$$\sin t = t-\frac{1}{3!}t^3+\frac{1}{5!}t^5-\ldots$$

we have:

$$\frac{\sin(3x)}{x^3} + \frac{r}{x^2} + s = \frac{\sin(3x) + rx + sx^3}{x^3} =$$

$$\frac{(3x - \frac{3^3}{3!}x^3 + \frac{3^5}{5!}x^5 - \ldots) + rx + sx^3}{x^3} =$$

$$\frac{(3 + r)x + \left(s - \frac{3^3}{3!}\right)x^3 + \frac{3^5}{5!}x^5 - \ldots}{x^3}$$

If this limit equals 0, then 3 + r = 0 and s – 3^3/(3!) = 0.

Hence r = -3 and s = 9/2