WORKSHEET X

SEQUENCES

1. Explain precisely what it means for a sequence $\{a_n\}$ to *converge*. What does it mean to say that a sequence *diverges*? What is meant by the *limit* of a sequence?

2. Discuss the rules for convergence (and divergence) of a sum, difference, product, and quotient of two sequences. State the Squeeze (or Sandwich) Theorem for sequences. What is a *monotonic sequence*? What can be said about an increasing sequence that is bounded above? Is every bounded sequence convergent? Is every convergent sequence bounded?

3. Explain why the limit of a convergent sequence must be unique.

4. For each of the following sequences, $\{t_n\}$, determine convergence or divergence. If the sequence converges, are you able to find its limit?

(a) $a_n = (-1)^n$

(b) $t_n = \sin(\pi n/2)$

(c) $t_n = \cos(\pi/n)$

(d) $t_n = (\ln n) / n$

(e) $t_n = 3 \arctan(n^2)$

(f) $t_n = (\ln n) / (\ln \ln n)$

(g) $t_n = (1 + 1/n)^n$

(h) $t_n = (-1)^{n+1} n^2$

(i) $t_n = ½ + (-1)^n/2$

(j) $t_n = n! / (n+3)$

(k) $t_n = (\cosh n) / (\sinh n)$

(l) $t_n = 1/2 + 1/3 + 1/4 + ... + 1/n$

(m) $t_n = (n^4 – 3n^2 + n^5 + 13) / (n^3 \ln n – 5n^5 + \ln(1+n^6) – 99)$

(n) $t_n = (\sin n) / n$

(o) $t_n = 1/n!$

(p) $t_n = 2^n / n!$

(q) $t_n = n! / 2^n$

(r) $t_n = (n^2 + 1)^{1/2} – n$

(s) $t_n = (n^2 + n + 1)^{1/2} – n$

(t) $t_n = (n^2 + 5n + 1)^{1/2} – (n^2 + n + 1)^{1/2}$

(u) $t_n = n \sin(1/n)$

(v) $t_n = \ln(n+1) – \ln n$

(w) $t_n = \ln(n^3+n+1) – \ln(n^2–n+5)$

(x) $t_n = n^{1/n}$

(y) $t_n = (1 + 3/n)^n$

(z) $b_n = n! / n^n$

5. For each of the following *recursively defined sequences*, do you believe that it converges or diverges? Give evidence. In the former case, assume that the limit exists and find it.

(a) $t_1 = 1$, $t_2 = 1$, $t_n = t_{n-1} + t_{n-2}$ for all $n \geq 3$ (Of what value is the Fibonacci sequence?)

(b) $a_1 = 4$, $a_n = a_{n-1}/2$ for all $n \geq 2$

(c) $c_1 = 3$, $c_n = 1.01(c_{n-1})$ for all $n \geq 2$

(d) $b_1 = 1$, $b_n = (b_{n-1} + 3/b_{n-1})/2$ for all $n \geq 2$

(e) $h_1 = 1$, $h_n = h_{n-1} + 1/n$ for all $n \geq 2$

(f) $a_1 = 1$, $a_2 = 2$, $a_n = (a_{n-1} + a_{n-2})/2$ for all $n \geq 3$.

(g) $z_1 = 1/3$, $z_n = (z_{n-1})^2$ for all $n \geq 2$.

6. Suppose that a sequence $\{x_n\}$ is defined recursively by:

$$x_0 = 1,\ x_1 = 2, \text{ and } x_{n+1} = 3x_n + x_{n-1}$$

Assuming that the limit of x_{n+1}/x_n exists, find it.

Mathematicians have tried in vain to this day to discover some order in the sequence of prime numbers, and we have reason to believe that it is a mystery into which the human mind will never penetrate.

- Leonhard Euler (1707-1783)

At 6 P.M. the well marked 1/2 inch of water, at nightfall 3/4 and at daybreak 7/8 of an inch. By noon of the next day there was 15/16 and on the next night 31/32 of an inch of water in the hold. The situation was desperate. At this rate of increase few, if any, could tell where it would rise to in a few days.

- Stephen Leacock