

WORKSHEET VII

HYPERBOLIC FUNCTIONS



The St. Louis arch is in the shape of a hyperbolic cosine.

Hyperbolic functions are very useful in both mathematics and physics. You may have already encountered them in Math 118 or in Math 161. If not, here are their definitions:

$$\sinh x = (e^x - e^{-x})/2$$

$$\cosh x = (e^x + e^{-x})/2$$

$$\tanh x = \sinh(x) / \cosh(x)$$

$$\coth x = 1/\tanh(x)$$

$$\operatorname{sech} x = 1/\cosh(x)$$

$$\operatorname{csch} x = 1/\sinh(x)$$

Oddly enough, they enjoy certain similarities with the trigonometric functions, with which you are much more familiar.

1. Graph the six hyperbolic functions: $\sinh x$, $\cosh x$, $\tanh x$, $\coth x$, $\operatorname{sech} x$, $\operatorname{csch} x$. For each curve, determine the limit of y as x tends toward infinity or negative infinity. Which of the

functions are *odd*? which are *even*? (Remember that an *odd function* is one that is symmetric with respect to the origin; an *even function* is one that is symmetric with respect to the y-axis.)

2. Find the derivative and the indefinite integral of each of the six hyperbolic functions.

3. Expand $\cosh(x+y)$, $\cosh(2x)$, $\tanh(x+y)$, and $\tanh(2x)$.

4. Show that $(\cosh x)^2 - (\sinh x)^2 = 1$.

5. Show that $1 - (\tanh x)^2 = (\operatorname{sech} x)^2$.

6. Show that:

$$\cosh \frac{x}{2} = \sqrt{\frac{\cosh x + 1}{2}}$$

(Note that this corresponds to the half-angle formula for cosine. Similar formulas exist for $\sinh(x/2)$ and $\tanh(x/2)$.)

(*Hint:* Compare the squares of each of the two sides.)

7. Find the limit of $(\sinh x) / e^x$ as x tends toward infinity.

8. Simplify the expression:

$$\sinh\left(\ln\left(x + \sqrt{x^2 + 1}\right)\right)$$

Use your answer to find a formula for the inverse of $\sinh(x)$.

9. The inverse of $\sinh x$ in Mathematica is represented by $\operatorname{ArcSinh}[x]$. Graph the curve $y = \operatorname{ArcSinh}(x)$. Find formulas for the derivative and the integral of $\operatorname{arcsinh}(x)$.

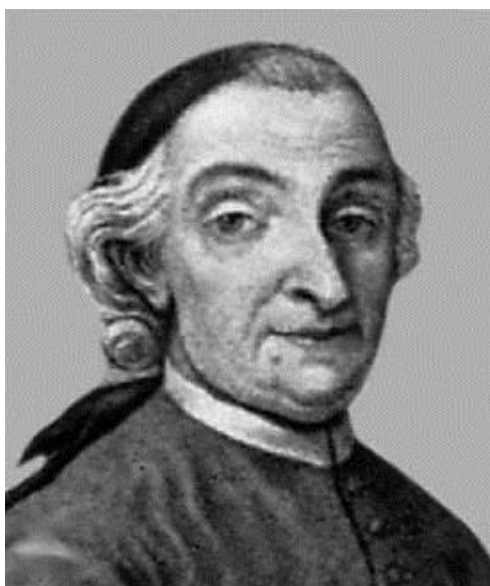
10. Repeat question 9 for the functions $\operatorname{ArcCosh}(x)$ and $\operatorname{ArcTanh}(x)$.

11. If the ends of a chain are attached to the points $(-1, 0)$ and $(1, 0)$, the chain will take the shape of the curve (called a *catenary*) given by:

$$f(x) = \frac{\cosh(ax) - \cosh a}{a}$$

where the constant a depends upon the length of the chain.

Show that for any value of a , the graph of $y = f(x)$ passes through the two points $(-1, 0)$ and $(1, 0)$.



[Vincenzo Riccati](#) (1707 - 1775) is given credit for introducing the hyperbolic functions.