MATH 116 - TEAM HOMEWORK 5
WINTER 2016

(1) O-guk is now good friends with all his previous opponents, especially with Junior and Prince V, to whom he has shown the path to benevolence. Little does he know that he will not succeed in doing that with his last enemy. O-guk and his friends have traveled to a distant planet, in order to stop the evil emperor of the universe, Lord Refreeze. Over the last few years, he has brought terror to many worlds. This battle is for the fate of the universe! First, Junior tries to stop the refrigerator-looking monster. Junior's power is given by

$$\sum_{n=1}^{\infty} \ln \frac{\arctan(n+1)}{\arctan(n)}$$

Refreeze's power is given by

$$\sum_{n=1}^{\infty} \frac{1}{3^{a_n}}$$

where the sequence (a_n) is defined recursively by

$$a_{n+1} = a_n + 2, a_1 = -1$$

Compute the sum of the two series to determine the winner of this fight. To compute the first sum, two things that you could try are to write the first few terms of the series and use properties of logs. To compute the second sum you should first try to find a formula for the sequence (a_n).

(2) Now it is Prince V's turn to take up the challenge. He wants to avenge his people: one of the world's Refreeze has destroyed was his home planet, the planet of the monkeys, where he and O-guk were born. When the fight begins, Prince V and Refreeze throw infinite series to each other. The one who throws the largest number of convergent series onto his opponent wins the fight. Prince V throws the series

$$(a)\ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+1}$$

$$(b)\ \sum_{n=1}^{\infty} \frac{1+2^n}{n+2^n}$$

$$(c)\ \sum_{n=1}^{\infty} \frac{a_n}{a_n+1}$$

where the series $\sum_{n=1}^{\infty}(a_n - 5)$ is known to converge.

Refreeze throws the series

$$(i)\ \sum_{n=0}^{\infty} b_n 3^n$$

when it is known that the power series $\sum_{n=0}^{\infty} b_n(x+5)^n$ has radius of convergence $R = 4$.

$$(ii)\ \sum_{n=1}^{\infty}(c_n^2 + 1)$$

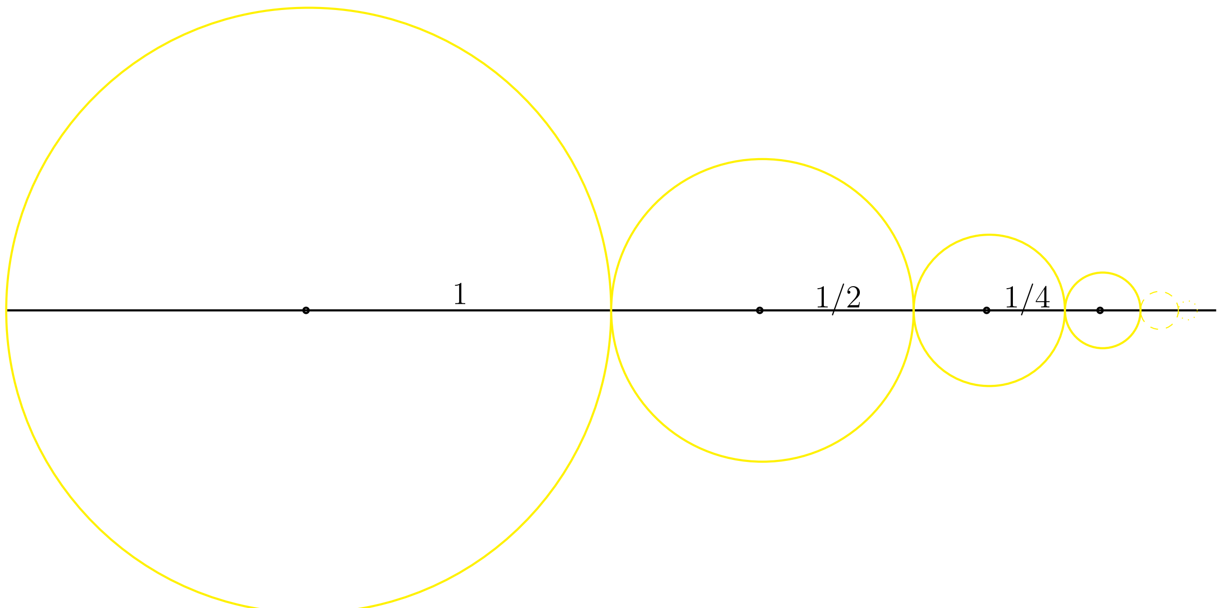
when it is known that the series $\sum_{n=1}^{\infty} c_n$ converges.

$$(iii)\ \sum_{n=1}^{\infty}(d_n^2 - 1)$$

where $1 \leq d_n \leq 1 + \frac{1}{n^2}$.

Determine the convergence of all the series, giving full justification, to find out who wins the fight.

(3) Finally, O-guk steps up to fight. He powers up to maximum and his power is given by the sum of the areas of the circles in the following figure. The circles have radii $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...$



Refreeze's power is now given by the perimeter of the first 10 of those circles.

Compute these two quantities exactly and determine the outcome of the third fight.

(4) After Refreeze defeats all the fighters, it seems that all hope is lost. To make it worse, he says that in those fights he only used one percent of his power. In a demonstration of his real powers, Refreeze attacked Pillin, O-guk's best friend since childhood. This was his last mistake! After seeing his best friend attacked, O-guk's anger towards Refreeze makes his powers to reach a new level. It is now given by the series

$$\sum_{n=2}^{\infty} \ln \frac{1}{1 - \frac{1}{\sqrt{n}}}$$

Determine the convergence of this series to see how powerful O-guk has become. (Hint: For one solution, show that for $0 \leq x < 1$, $x \leq \ln \dfrac{1}{1-x}$)