Groupwork VII   problems from Stewart

11.1 EXERCISES

1. (a) What is a sequence?
   (b) What does it mean to say that $\lim_{n\to\infty} a_n = 8$?
   (c) What does it mean to say that $\lim_{n\to\infty} a_n = \infty$?

2. (a) What is a convergent sequence? Give two examples.
   (b) What is a divergent sequence? Give two examples.

3–8 List the first five terms of the sequence.

3. $a_n = 1 - (0.2)^n$

4. $a_n = \dfrac{n+1}{3n-1}$

5. $a_n = \dfrac{3(-1)^n}{n!}$

6. $\{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)\}$

7. $a_1 = 3, \quad a_{n+1} = 2a_n - 1$

8. $a_1 = 4, \quad a_{n+1} = \dfrac{a_n}{a_n - 1}$

9–14 Find a formula for the general term a_n of the sequence, assuming that the pattern of the first few terms continues.

9. $\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots\}$

10. $\{1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots\}$

11. $\{2, 7, 12, 17, \ldots\}$

12. $\{-\frac{1}{4}, \frac{2}{9}, -\frac{3}{16}, \frac{4}{25}, \ldots\}$

13. $\{1, -\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \ldots\}$

14. $\{5, 1, 5, 1, 5, 1, \ldots\}$

15. List the first six terms of the sequence defined by

$$a_n = \frac{n}{2n+1}$$

Does the sequence appear to have a limit? If so, find it.

16. List the first nine terms of the sequence $\{\cos(n\pi/3)\}$. Does this sequence appear to have a limit? If so, find it. If not, explain why.

17–46 Determine whether the sequence converges or diverges. If it converges, find the limit.

17. $a_n = 1 - (0.2)^n$

18. $a_n = \dfrac{n^3}{n^3 + 1}$

19. $a_n = \dfrac{3 + 5n^2}{n + n^2}$

20. $a_n = \dfrac{n^3}{n + 1}$

21. $a_n = e^{1/n}$

22. $a_n = \dfrac{3^{n+2}}{5^n}$

23. $a_n = \tan\left(\dfrac{2n\pi}{1 + 8n}\right)$

24. $a_n = \sqrt{\dfrac{n + 1}{9n + 1}}$

25. $a_n = \dfrac{(-1)^{n-1}n}{n^2 + 1}$

26. $a_n = \dfrac{(-1)^n n^3}{n^3 + 2n^2 + 1}$

27. $a_n = \cos(n/2)$

28. $a_n = \cos(2/n)$

29. $\left\{\dfrac{(2n - 1)!}{(2n + 1)!}\right\}$

30. $\{\arctan 2n\}$

31. $\left\{\dfrac{e^n + e^{-n}}{e^{2n} - 1}\right\}$

32. $\left\{\dfrac{\ln n}{\ln 2n}\right\}$

33. $\{n^2 e^{-n}\}$

34. $\{n \cos n\pi\}$

35. $a_n = \dfrac{\cos^2 n}{2^n}$

36. $a_n = \ln(n + 1) - \ln n$

37. $a_n = n \sin(1/n)$

38. $a_n = \sqrt[n]{2^{1+3n}}$

39. $a_n = \left(1 + \dfrac{2}{n}\right)^n$

40. $a_n = \dfrac{\sin 2n}{1 + \sqrt{n}}$

41. $a_n = \ln(2n^2 + 1) - \ln(n^2 + 1)$

42. $a_n = \dfrac{(\ln n)^2}{n}$

43. $\{0, 1, 0, 0, 1, 0, 0, 0, 1, \dots\}$

44. $\left\{\frac{1}{1}, \frac{1}{3}, \frac{1}{2}, \frac{1}{4}, \frac{1}{3}, \frac{1}{5}, \frac{1}{4}, \frac{1}{6}, \dots\right\}$

45. $a_n = \dfrac{n!}{2^n}$

46. $a_n = \dfrac{(-3)^n}{n!}$

47–53 Use a graph of the sequence to decide whether the sequence is convergent or divergent. If the sequence is convergent, guess the value of the limit from the graph and then prove your guess. (See the margin note on page 680 for advice on graphing sequences.)

47. $a_n = 1 + (-2/e)^n$

48. $a_n = \sqrt{n}\,\sin\left(\pi/\sqrt{n}\right)$

49. $a_n = \sqrt{\dfrac{3 + 2n^2}{8n^2 + n}}$

50. $a_n = \sqrt[n]{3^n + 5^n}$

51. $a_n = \dfrac{n^2 \cos n}{1 + n^2}$

52. $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n - 1)}{n!}$

53. $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n - 1)}{(2n)^n}$

54. (a) Determine whether the sequence defined as follows is convergent or divergent:

$$a_1 = 1 \qquad a_{n+1} = 4 - a_n \qquad \text{for } n \geq 1$$

(b) What happens if the first term is $a_1 = 2$?

55. If \$1000 is invested at 6% interest, compounded annually, then after n years the investment is worth $a_n = 1000(1.06)^n$ dollars.
(a) Find the first five terms of the sequence $\{a_n\}$.
(b) Is the sequence convergent or divergent? Explain.

56. Find the first 40 terms of the sequence defined by

$$a_{n+1} = \begin{cases} \frac{1}{2}a_n & \text{if } a_n \text{ is an even number} \\ 3a_n + 1 & \text{if } a_n \text{ is an odd number} \end{cases}$$

and $a_1 = 11$. Do the same if $a_1 = 25$. Make a conjecture about this type of sequence.

57. For what values of r is the sequence $\{nr^n\}$ convergent?

58. (a) If $\{a_n\}$ is convergent, show that

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n$$

(b) A sequence $\{a_n\}$ is defined by $a_1 = 1$ and $a_{n+1} = 1/(1 + a_n)$ for $n \geq 1$. Assuming that $\{a_n\}$ is convergent, find its limit.

59. Suppose you know that $\{a_n\}$ is a decreasing sequence and all its terms lie between the numbers 5 and 8. Explain why the sequence has a limit. What can you say about the value of the limit?

60–66 Determine whether the sequence is increasing, decreasing, or not monotonic. Is the sequence bounded?

60. $a_n = (-2)^{n+1}$

61. $a_n = \dfrac{1}{2n + 3}$

62. $a_n = \dfrac{2n - 3}{3n + 4}$

63. $a_n = n(-1)^n$

64. $a_n = ne^{-n}$

65. $a_n = \dfrac{n}{n^2 + 1}$

66. $a_n = n + \dfrac{1}{n}$

67. Find the limit of the sequence

$$\left\{\sqrt{2},\ \sqrt{2\sqrt{2}},\ \sqrt{2\sqrt{2\sqrt{2}}},\ \dots\right\}$$

68. A sequence $\{a_n\}$ is given by $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$.
(a) By induction or otherwise, show that $\{a_n\}$ is increasing and bounded above by 3. Apply the Monotonic Sequence Theorem to show that $\lim_{n\to\infty} a_n$ exists.
(b) Find $\lim_{n\to\infty} a_n$.

69. Show that the sequence defined by

$$a_1 = 1 \qquad a_{n+1} = 3 - \frac{1}{a_n}$$

is increasing and $a_n < 3$ for all n. Deduce that $\{a_n\}$ is convergent and find its limit.

70. Show that the sequence defined by

$$a_1 = 2 \qquad a_{n+1} = \frac{1}{3 - a_n}$$

satisfies $0 < a_n \leqslant 2$ and is decreasing. Deduce that the sequence is convergent and find its limit.

71. (a) Fibonacci posed the following problem: Suppose that rabbits live forever and that every month each pair produces a new pair which becomes productive at age 2 months. If we start with one newborn pair, how many pairs of rabbits will we have in the nth month? Show that the answer is f_n, where $\{f_n\}$ is the Fibonacci sequence defined in Example 3(c).
(b) Let $a_n = f_{n+1}/f_n$ and show that $a_{n-1} = 1 + 1/a_{n-2}$. Assuming that $\{a_n\}$ is convergent, find its limit.

72. (a) Let $a_1 = a$, $a_2 = f(a)$, $a_3 = f(a_2) = f(f(a)), \ldots,$ $a_{n+1} = f(a_n)$, where f is a continuous function. If $\lim_{n\to\infty} a_n = L$, show that $f(L) = L$.
(b) Illustrate part (a) by taking $f(x) = \cos x$, $a = 1$, and estimating the value of L to five decimal places.

73. (a) Use a graph to guess the value of the limit

$$\lim_{n\to\infty} \frac{n^5}{n!}$$

(b) Use a graph of the sequence in part (a) to find the smallest values of N that correspond to $\varepsilon = 0.1$ and $\varepsilon = 0.001$ in Definition 2.

74. Use Definition 2 directly to prove that $\lim_{n\to\infty} r^n = 0$ when $|r| < 1$.

75. Prove Theorem 6.
[*Hint:* Use either Definition 2 or the Squeeze Theorem.]

76. Prove Theorem 7.

77. Prove that if $\lim_{n\to\infty} a_n = 0$ and $\{b_n\}$ is bounded, then $\lim_{n\to\infty} (a_n b_n) = 0$.

78. Let $a_n = \left(1 + \dfrac{1}{n}\right)^n$.
(a) Show that if $0 \leqslant a < b$, then

$$\frac{b^{n+1} - a^{n+1}}{b - a} < (n + 1)b^n$$

(b) Deduce that $b^n[(n + 1)a - nb] < a^{n+1}$.
(c) Use $a = 1 + 1/(n + 1)$ and $b = 1 + 1/n$ in part (b) to show that $\{a_n\}$ is increasing.
(d) Use $a = 1$ and $b = 1 + 1/(2n)$ in part (b) to show that $a_{2n} < 4$.
(e) Use parts (c) and (d) to show that $a_n < 4$ for all n.
(f) Use Theorem 12 to show that $\lim_{n\to\infty} (1 + 1/n)^n$ exists. (The limit is e. See Equation 3.6.6.)

79. Let a and b be positive numbers with $a > b$. Let a_1 be their arithmetic mean and b_1 their geometric mean:

$$a_1 = \frac{a + b}{2} \qquad b_1 = \sqrt{ab}$$

Repeat this process so that, in general,

$$a_{n+1} = \frac{a_n + b_n}{2} \qquad b_{n+1} = \sqrt{a_n b_n}$$

(a) Use mathematical induction to show that

$$a_n > a_{n+1} > b_{n+1} > b_n$$

(b) Deduce that both $\{a_n\}$ and $\{b_n\}$ are convergent.
(c) Show that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. Gauss called the common value of these limits the **arithmetic-geometric mean** of the numbers a and b.

80. (a) Show that if $\lim_{n\to\infty} a_{2n} = L$ and $\lim_{n\to\infty} a_{2n+1} = L$, then $\{a_n\}$ is convergent and $\lim_{n\to\infty} a_n = L$.
(b) If $a_1 = 1$ and

$$a_{n+1} = 1 + \frac{1}{1 + a_n}$$

find the first eight terms of the sequence $\{a_n\}$. Then use part (a) to show that $\lim_{n\to\infty} a_n = \sqrt{2}$. This gives the **continued fraction expansion**

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}$$

81. The size of an undisturbed fish population has been modeled by the formula

$$p_{n+1} = \frac{bp_n}{a + p_n}$$

where p_n is the fish population after n years and a and b are positive constants that depend on the species and its environment. Suppose that the population in year 0 is $p_0 > 0$.
(a) Show that if $\{p_n\}$ is convergent, then the only possible values for its limit are 0 and $b - a$.
(b) Show that $p_{n+1} < (b/a)p_n$.
(c) Use part (b) to show that if $a > b$, then $\lim_{n\to\infty} p_n = 0$; in other words, the population dies out.
(d) Now assume that $a < b$. Show that if $p_0 < b - a$, then $\{p_n\}$ is increasing and $0 < p_n < b - a$. Show also that if $p_0 > b - a$, then $\{p_n\}$ is decreasing and $p_n > b - a$. Deduce that if $a < b$, then $\lim_{n\to\infty} p_n = b - a$.

11.2 EXERCISES

1. (a) What is the difference between a sequence and a series?
   (b) What is a convergent series? What is a divergent series?

2. Explain what it means to say that $\sum_{n=1}^{\infty} a_n = 5$.

3–8 Find at least 10 partial sums of the series. Graph both the sequence of terms and the sequence of partial sums on the same screen. Does it appear that the series is convergent or divergent? If it is convergent, find the sum. If it is divergent, explain why.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{12}{(-5)^n}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2 - 1}{n^2 + 1}$

5. $\displaystyle\sum_{n=1}^{\infty} \tan n$

6. $\displaystyle\sum_{n=1}^{\infty} (0.6)^{n-1}$

7. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$

8. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(n+2)}$

9. Let $a_n = \dfrac{2n}{3n+1}$.
   (a) Determine whether $\{a_n\}$ is convergent.
   (b) Determine whether $\sum_{n=1}^{\infty} a_n$ is convergent.

10. (a) Explain the difference between

$$\sum_{i=1}^{n} a_i \quad \text{and} \quad \sum_{j=1}^{n} a_j$$

   (b) Explain the difference between

$$\sum_{i=1}^{n} a_i \quad \text{and} \quad \sum_{i=1}^{n} a_j$$

11–20 Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum.

11. $3 + 2 + \frac{4}{3} + \frac{8}{9} + \cdots$

12. $\frac{1}{8} - \frac{1}{4} + \frac{1}{2} - 1 + \cdots$

13. $3 - 4 + \frac{16}{3} - \frac{64}{9} + \cdots$

14. $1 + 0.4 + 0.16 + 0.064 + \cdots$

15. $\displaystyle\sum_{n=1}^{\infty} 6(0.9)^{n-1}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{(-9)^{n-1}}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$

18. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(\sqrt{2})^n}$

19. $\displaystyle\sum_{n=0}^{\infty} \frac{\pi^n}{3^{n+1}}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}}$

21–34 Determine whether the series is convergent or divergent. If it is convergent, find its sum.

21. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{2n-3}$

23. $\displaystyle\sum_{k=2}^{\infty} \frac{k^2}{k^2-1}$

24. $\displaystyle\sum_{k=1}^{\infty} \frac{k(k+2)}{(k+3)^2}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{1+2^n}{3^n}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{1+3^n}{2^n}$

27. $\displaystyle\sum_{n=1}^{\infty} \sqrt[n]{2}$

28. $\displaystyle\sum_{n=1}^{\infty} [(0.8)^{n-1} - (0.3)^n]$

29. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n^2+1}{2n^2+1}\right)$

30. $\displaystyle\sum_{k=1}^{\infty} (\cos 1)^k$

31. $\displaystyle\sum_{n=1}^{\infty} \arctan n$

32. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{2}{n}\right)$

33. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{e^n} + \frac{1}{n(n+1)}\right)$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^2}$

35–40 Determine whether the series is convergent or divergent by expressing s_n as a telescoping sum (as in Example 6). If it is convergent, find its sum.

35. $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n^2-1}$

36. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3}$

37. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$

38. $\displaystyle\sum_{n=1}^{\infty} \ln \frac{n}{n+1}$

39. $\sum_{n=1}^{\infty} \left(e^{1/n} - e^{1/(n+1)}\right)$

40. $\sum_{n=1}^{\infty} \left(\cos\frac{1}{n^2} - \cos\frac{1}{(n+1)^2}\right)$

41–46 Express the number as a ratio of integers.

41. $0.\overline{2} = 0.2222\ldots$

42. $0.\overline{73} = 0.73737373\ldots$

43. $3.\overline{417} = 3.417417417\ldots$

44. $6.2\overline{54} = 6.2545454\ldots$

45. $1.53\overline{42}$

46. $7.\overline{12345}$

47–51 Find the values of x for which the series converges. Find the sum of the series for those values of x.

47. $\sum_{n=1}^{\infty} \frac{x^n}{3^n}$

48. $\sum_{n=1}^{\infty} (x-4)^n$

49. $\sum_{n=0}^{\infty} 4^n x^n$

50. $\sum_{n=0}^{\infty} \frac{(x+3)^n}{2^n}$

51. $\sum_{n=0}^{\infty} \frac{\cos^n x}{2^n}$

52. We have seen that the harmonic series is a divergent series whose terms approach 0. Show that

$$\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right)$$

is another series with this property.

CAS 53–54 Use the partial fraction command on your CAS to find a convenient expression for the partial sum, and then use this expression to find the sum of the series. Check your answer by using the CAS to sum the series directly.

53. $\sum_{n=1}^{\infty} \frac{3n^2 + 3n + 1}{(n^2 + n)^3}$

54. $\sum_{n=2}^{\infty} \frac{1}{n^3 - n}$

55. If the nth partial sum of a series $\sum_{n=1}^{\infty} a_n$ is

$$s_n = \frac{n-1}{n+1}$$

find a_n and $\sum_{n=1}^{\infty} a_n$.

56. If the nth partial sum of a series $\sum_{n=1}^{\infty} a_n$ is $s_n = 3 - n2^{-n}$, find a_n and $\sum_{n=1}^{\infty} a_n$.

57. When money is spent on goods and services, those who receive the money also spend some of it. The people receiving some of the twice-spent money will spend some of that, and so on. Economists call this chain reaction the *multiplier effect*. In a hypothetical isolated community, the local government begins the process by spending D dollars. Suppose that each recipient of spent money spends $100c\%$ and saves $100s\%$ of the money that he or she receives. The values c and s are called the *marginal propensity to consume* and the *marginal propensity to save* and, of course, $c + s = 1$.

(a) Let S_n be the total spending that has been generated after n transactions. Find an equation for S_n.

(b) Show that $\lim_{n\to\infty} S_n = kD$, where $k = 1/s$. The number k is called the *multiplier*. What is the multiplier if the marginal propensity to consume is 80%?

Note: The federal government uses this principle to justify deficit spending. Banks use this principle to justify lending a large percentage of the money that they receive in deposits.

58. A certain ball has the property that each time it falls from a height h onto a hard, level surface, it rebounds to a height rh, where $0 < r < 1$. Suppose that the ball is dropped from an initial height of H meters.

(a) Assuming that the ball continues to bounce indefinitely, find the total distance that it travels. (Use the fact that the ball falls $\frac{1}{2}gt^2$ meters in t seconds.)

(b) Calculate the total time that the ball travels.

(c) Suppose that each time the ball strikes the surface with velocity v it rebounds with velocity $-kv$, where $0 < k < 1$. How long will it take for the ball to come to rest?

59. Find the value of c if

$$\sum_{n=2}^{\infty} (1 + c)^{-n} = 2$$

60. Find the value of c such that

$$\sum_{n=0}^{\infty} e^{nc} = 10$$

61. In Example 7 we showed that the harmonic series is divergent. Here we outline another method, making use of the fact that $e^x > 1 + x$ for any $x > 0$. (See Exercise 4.3.76.)

If s_n is the nth partial sum of the harmonic series, show that $e^{s_n} > n + 1$. Why does this imply that the harmonic series is divergent?

62. Graph the curves $y = x^n$, $0 \leq x \leq 1$, for $n = 0, 1, 2, 3, 4, \ldots$ on a common screen. By finding the areas between successive curves, give a geometric demonstration of the fact, shown in Example 6, that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

11.3 EXERCISES

1. Draw a picture to show that

$$\sum_{n=2}^{\infty} \frac{1}{n^{1.3}} < \int_{1}^{\infty} \frac{1}{x^{1.3}}\,dx$$

What can you conclude about the series?

2. Suppose f is a continuous positive decreasing function for $x \geqslant 1$ and $a_n = f(n)$. By drawing a picture, rank the following three quantities in increasing order:

$$\int_{1}^{6} f(x)\,dx \qquad \sum_{i=1}^{5} a_i \qquad \sum_{i=2}^{6} a_i$$

3–8 Use the Integral Test to determine whether the series is convergent or divergent.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n+1)^3}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+4}}$

7. $\displaystyle\sum_{n=1}^{\infty} ne^{-n}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{n+2}{n+1}$

9–26 Determine whether the series is convergent or divergent.

9. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^{0.85}}$

10. $\displaystyle\sum_{n=1}^{\infty} (n^{-1.4} + 3n^{-1.2})$

11. $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \cdots$

12. $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \cdots$

13. $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \cdots$

14. $\frac{1}{5} + \frac{1}{8} + \frac{1}{11} + \frac{1}{14} + \frac{1}{17} + \cdots$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{5 - 2\sqrt{n}}{n^3}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + 4}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{3n + 2}{n(n + 1)}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 - 4n + 5}$

21. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

22. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$

24. $\displaystyle\sum_{n=3}^{\infty} \frac{n^2}{e^n}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3 + n}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4 + 1}$

27–30 Find the values of p for which the series is convergent.

27. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$

28. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \ln n\,[\ln(\ln n)]^p}$

29. $\displaystyle\sum_{n=1}^{\infty} n(1 + n^2)^p$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$

31. The Riemann zeta-function ζ is defined by

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$$

and is used in number theory to study the distribution of prime numbers. What is the domain of ζ?

32. (a) Find the partial sum s_{10} of the series $\sum_{n=1}^{\infty} 1/n^4$. Estimate the error in using s_{10} as an approximation to the sum of the series.
(b) Use (3) with $n = 10$ to give an improved estimate of the sum.
(c) Find a value of n so that s_n is within 0.00001 of the sum.

33. (a) Use the sum of the first 10 terms to estimate the sum of the series $\sum_{n=1}^{\infty} 1/n^2$. How good is this estimate?
(b) Improve this estimate using (3) with $n = 10$.
(c) Find a value of n that will ensure that the error in the approximation $s \approx s_n$ is less than 0.001.

34. Find the sum of the series $\sum_{n=1}^{\infty} 1/n^5$ correct to three decimal places.

35. Estimate $\sum_{n=1}^{\infty} (2n + 1)^{-6}$ correct to five decimal places.

36. How many terms of the series $\sum_{n=2}^{\infty} 1/[n(\ln n)^2]$ would you need to add to find its sum to within 0.01?

37. Show that if we want to approximate the sum of the series $\sum_{n=1}^{\infty} n^{-1.001}$ so that the error is less than 5 in the ninth decimal place, then we need to add more than $10^{11,301}$ terms!

CAS **38.** (a) Show that the series $\sum_{n=1}^{\infty} (\ln n)^2/n^2$ is convergent.
(b) Find an upper bound for the error in the approximation $s \approx s_n$.
(c) What is the smallest value of n such that this upper bound is less than 0.05?
(d) Find s_n for this value of n.

39. (a) Use (4) to show that if s_n is the nth partial sum of the harmonic series, then

$$s_n \leq 1 + \ln n$$

(b) The harmonic series diverges, but very slowly. Use part (a) to show that the sum of the first million terms is less than 15 and the sum of the first billion terms is less than 22.

40. Use the following steps to show that the sequence

$$t_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n$$

has a limit. (The value of the limit is denoted by γ and is called Euler's constant.)
(a) Draw a picture like Figure 6 with $f(x) = 1/x$ and interpret t_n as an area [or use (5)] to show that $t_n > 0$ for all n.
(b) Interpret

$$t_n - t_{n+1} = [\ln(n + 1) - \ln n] - \frac{1}{n + 1}$$

as a difference of areas to show that $t_n - t_{n+1} > 0$. Therefore, $\{t_n\}$ is a decreasing sequence.
(c) Use the Monotonic Sequence Theorem to show that $\{t_n\}$ is convergent.

41. Find all positive values of b for which the series $\sum_{n=1}^{\infty} b^{\ln n}$ converges.

42. Find all values of c for which the following series converges.

$$\sum_{n=1}^{\infty} \left(\frac{c}{n} - \frac{1}{n + 1} \right)$$

11.4 EXERCISES

1. Suppose Σa_n and Σb_n are series with positive terms and Σb_n is known to be convergent.
   (a) If $a_n > b_n$ for all n, what can you say about Σa_n? Why?
   (b) If $a_n < b_n$ for all n, what can you say about Σa_n? Why?

2. Suppose Σa_n and Σb_n are series with positive terms and Σb_n is known to be divergent.
   (a) If $a_n > b_n$ for all n, what can you say about Σa_n? Why?
   (b) If $a_n < b_n$ for all n, what can you say about Σa_n? Why?

3–32 Determine whether the series converges or diverges.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n^3+1}$

4. $\displaystyle\sum_{n=2}^{\infty} \frac{n^3}{n^4-1}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^2\sqrt{n}}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{4+3^n}{2^n}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2+1}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-1}{3n^4+1}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n4^n}$

12. $\displaystyle\sum_{n=0}^{\infty} \frac{1+\sin n}{10^n}$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{n^{1.2}}$

14. $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{2+(-1)^n}{n\sqrt{n}}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n+3}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{1+4^n}{1+3^n}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{n+4^n}{n+6^n}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+2}}{2n^2+n+1}$

22. $\displaystyle\sum_{n=3}^{\infty} \frac{n+2}{(n+1)^3}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{5+2n}{(1+n^2)^2}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-5n}{n^3+n+1}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{1+n+n^2}{\sqrt{1+n^2+n^6}}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{n+5}{\sqrt[3]{n^7+n^2}}$

27. $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^2 e^{-n}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

31. $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$

33–36 Use the sum of the first 10 terms to approximate the sum of the series. Estimate the error.

33. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^4+1}}$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^3}$

35. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+2^n}$

36. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n+1)3^n}$

37. The meaning of the decimal representation of a number $0.d_1d_2d_3\ldots$ (where the digit d_i is one of the numbers 0, 1, 2, . . . , 9) is that

$$0.d_1d_2d_3d_4\ldots = \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \frac{d_4}{10^4} + \cdots$$

Show that this series always converges.

38. For what values of p does the series $\Sigma_{n=2}^{\infty} 1/(n^p \ln n)$ converge?

39. Prove that if $a_n \geq 0$ and Σa_n converges, then Σa_n^2 also converges.

40. (a) Suppose that Σa_n and Σb_n are series with positive terms and Σb_n is convergent. Prove that if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0$$

then Σa_n is also convergent.
   (b) Use part (a) to show that the series converges.
   (i) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}\,e^n}$

41. (a) Suppose that Σa_n and Σb_n are series with positive terms and Σb_n is divergent. Prove that if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$$

then Σa_n is also divergent.
   (b) Use part (a) to show that the series diverges.
   (i) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$ (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$

42. Give an example of a pair of series Σa_n and Σb_n with positive terms where $\lim_{n\to\infty}(a_n/b_n) = 0$ and Σb_n diverges, but Σa_n converges. (Compare with Exercise 40.)

43. Show that if $a_n > 0$ and $\lim_{n\to\infty} na_n \neq 0$, then Σa_n is divergent.

44. Show that if $a_n > 0$ and Σa_n is convergent, then $\Sigma \ln(1+a_n)$ is convergent.

45. If Σa_n is a convergent series with positive terms, is it true that $\Sigma \sin(a_n)$ is also convergent?

46. If Σa_n and Σb_n are both convergent series with positive terms, is it true that $\Sigma a_n b_n$ is also convergent?

11.6 EXERCISES

1. What can you say about the series $\sum a_n$ in each of the following cases?

(a) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 8$ (b) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0.8$

(c) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$

2–28 Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

2. $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

3. $\sum_{n=0}^{\infty} \frac{(-10)^n}{n!}$

4. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^n}{n^4}$

5. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[4]{n}}$

6. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$

7. $\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^k$

8. $\sum_{n=1}^{\infty} \frac{n!}{100^n}$

9. $\sum_{n=1}^{\infty} (-1)^n \frac{(1.1)^n}{n^4}$

10. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3+2}}$

11. $\sum_{n=1}^{\infty} \frac{(-1)^n e^{1/n}}{n^3}$

12. $\sum_{n=1}^{\infty} \frac{\sin 4n}{4^n}$

13. $\sum_{n=1}^{\infty} \frac{10^n}{(n+1)4^{2n+1}}$

14. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2 2^n}{n!}$

15. $\sum_{n=1}^{\infty} \frac{(-1)^n \arctan n}{n^2}$

16. $\sum_{n=1}^{\infty} \frac{3 - \cos n}{n^{2/3} - 2}$

17. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

18. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

19. $\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n!}$

20. $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$

21. $\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$

22. $\sum_{n=2}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n}$

23. $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{n^2}$

24. $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$

25. $1 - \frac{1 \cdot 3}{3!} + \frac{1 \cdot 3 \cdot 5}{5!} - \frac{1 \cdot 3 \cdot 5 \cdot 7}{7!} + \cdots + (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n-1)!} + \cdots$

26. $\frac{2}{5} + \frac{2 \cdot 6}{5 \cdot 8} + \frac{2 \cdot 6 \cdot 10}{5 \cdot 8 \cdot 11} + \frac{2 \cdot 6 \cdot 10 \cdot 14}{5 \cdot 8 \cdot 11 \cdot 14} + \cdots$

27. $\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}{n!}$

28. $\sum_{n=1}^{\infty} (-1)^n \frac{2^n n!}{5 \cdot 8 \cdot 11 \cdot \cdots \cdot (3n+2)}$

29. The terms of a series are defined recursively by the equations

$$a_1 = 2 \qquad a_{n+1} = \frac{5n+1}{4n+3} a_n$$

Determine whether $\Sigma\, a_n$ converges or diverges.

30. A series $\Sigma\, a_n$ is defined by the equations

$$a_1 = 1 \qquad a_{n+1} = \frac{2 + \cos n}{\sqrt{n}} a_n$$

Determine whether $\Sigma\, a_n$ converges or diverges.

31. For which of the following series is the Ratio Test inconclusive (that is, it fails to give a definite answer)?

(a) $\sum_{n=1}^{\infty} \frac{1}{n^3}$ (b) $\sum_{n=1}^{\infty} \frac{n}{2^n}$

(c) $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{\sqrt{n}}$ (d) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^2}$

32. For which positive integers k is the following series convergent?

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(kn)!}$$

33. (a) Show that $\sum_{n=0}^{\infty} x^n/n!$ converges for all x.
(b) Deduce that $\lim_{n\to\infty} x^n/n! = 0$ for all x.

34. Let $\Sigma\, a_n$ be a series with positive terms and let $r_n = a_{n+1}/a_n$. Suppose that $\lim_{n\to\infty} r_n = L < 1$, so $\Sigma\, a_n$ converges by the Ratio Test. As usual, we let R_n be the remainder after n terms, that is,

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

(a) If $\{r_n\}$ is a decreasing sequence and $r_{n+1} < 1$, show, by summing a geometric series, that

$$R_n \leqslant \frac{a_{n+1}}{1 - r_{n+1}}$$

(b) If $\{r_n\}$ is an increasing sequence, show that

$$R_n \leqslant \frac{a_{n+1}}{1 - L}$$

35. (a) Find the partial sum s_5 of the series $\sum_{n=1}^{\infty} 1/n2^n$. Use Exercise 34 to estimate the error in using s_5 as an approximation to the sum of the series.
(b) Find a value of n so that s_n is within 0.00005 of the sum. Use this value of n to approximate the sum of the series.

36. Use the sum of the first 10 terms to approximate the sum of the series

$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

Use Exercise 34 to estimate the error.

37. Prove the Root Test. [*Hint for part (i):* Take any number r such that $L < r < 1$ and use the fact that there is an integer N such that $\sqrt[n]{|a_n|} < r$ whenever $n \geqslant N$.]

38. Around 1910, the Indian mathematician Srinivasa Ramanujan discovered the formula

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!(1103 + 26390n)}{(n!)^4 396^{4n}}$$

William Gosper used this series in 1985 to compute the first 17 million digits of π.
(a) Verify that the series is convergent.
(b) How many correct decimal places of π do you get if you use just the first term of the series? What if you use two terms?

39. Given any series $\Sigma\, a_n$, we define a series $\Sigma\, a_n^+$ whose terms are all the positive terms of $\Sigma\, a_n$ and a series $\Sigma\, a_n^-$ whose terms are all the negative terms of $\Sigma\, a_n$. To be specific, we let

$$a_n^+ = \frac{a_n + |a_n|}{2} \qquad a_n^- = \frac{a_n - |a_n|}{2}$$

Notice that if $a_n > 0$, then $a_n^+ = a_n$ and $a_n^- = 0$, whereas if $a_n < 0$, then $a_n^- = a_n$ and $a_n^+ = 0$.
(a) If $\Sigma\, a_n$ is absolutely convergent, show that both of the series $\Sigma\, a_n^+$ and $\Sigma\, a_n^-$ are convergent.
(b) If $\Sigma\, a_n$ is conditionally convergent, show that both of the series $\Sigma\, a_n^+$ and $\Sigma\, a_n^-$ are divergent.

40. Prove that if $\Sigma\, a_n$ is a conditionally convergent series and r is any real number, then there is a rearrangement of $\Sigma\, a_n$ whose sum is r. [*Hints:* Use the notation of Exercise 39. Take just enough positive terms a_n^+ so that their sum is greater than r. Then add just enough negative terms a_n^- so that the cumulative sum is less than r. Continue in this manner and use Theorem 11.2.6.]

11.7 EXERCISES

1–38 Test the series for convergence or divergence.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + 3^n}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n+1)^n}{n^{2n}}$

3. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$

4. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+2}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 2^{n-1}}{(-5)^n}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n+1}$

7. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

8. $\displaystyle\sum_{k=1}^{\infty} \frac{2^k k!}{(k+2)!}$

9. $\displaystyle\sum_{k=1}^{\infty} k^2 e^{-k}$

10. $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^3}$

11. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln n}$

12. $\displaystyle\sum_{n=1}^{\infty} \sin n$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin 2n}{1+2^n}$

15. $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n+2)}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}$

17. $\displaystyle\sum_{n=1}^{\infty} (-1)^n 2^{1/n}$

18. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$

19. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$

20. $\displaystyle\sum_{k=1}^{\infty} \frac{k+5}{5^k}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^{2n}}{n^n}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{n^3+2n^2+5}$

23. $\displaystyle\sum_{n=1}^{\infty} \tan(1/n)$

24. $\displaystyle\sum_{n=1}^{\infty} n \sin(1/n)$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{5^n}$

27. $\displaystyle\sum_{k=1}^{\infty} \frac{k \ln k}{(k+1)^3}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\cosh n}$

30. $\displaystyle\sum_{j=1}^{\infty} (-1)^j \frac{\sqrt{j}}{j+5}$

31. $\displaystyle\sum_{k=1}^{\infty} \frac{5^k}{3^k+4^k}$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$

33. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + n\cos^2 n}$

35. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

36. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$

37. $\displaystyle\sum_{n=1}^{\infty} (\sqrt[n]{2} - 1)^n$

38. $\displaystyle\sum_{n=1}^{\infty} (\sqrt[n]{2} - 1)$