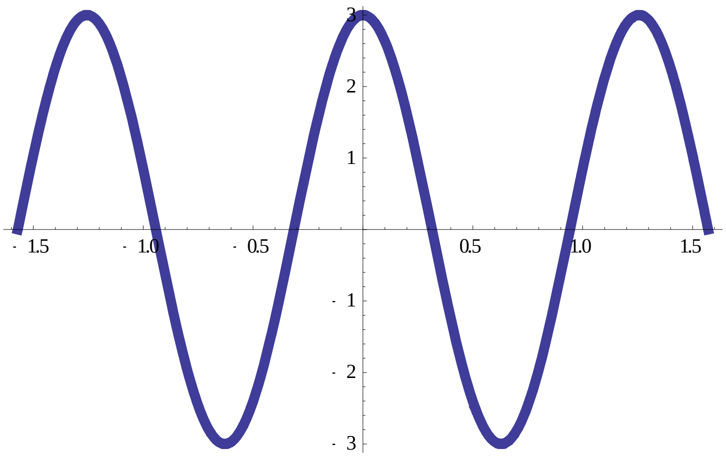
MATH 162 SOLUTIONS: QUIZ 1

1. *[5 pts]* Express the area under *one arch* of the curve $y = 3 \cos 5x$ as a Riemann (i.e. definite) integral. No need to evaluate this integral. *Sketch!*

Solution: First note that the period of cos 5x is (1/5)(period of cos x) = 2π/5.



So the area that we seek may be obtained by integrating from x = -2π/5 to 2π/5.

$$\textit{Thus area} = \int_{-\frac{\pi}{10}}^{\frac{\pi}{10}} 3\cos(5x)\, dx$$

2. *[5 pts]* Using the *Fundamental Theorem of Calculus*, compute the derivative of the function

$$F(x) = \int_1^x \frac{\ln t}{t^2}\, dt$$

Solution: Employing the FTC, we find

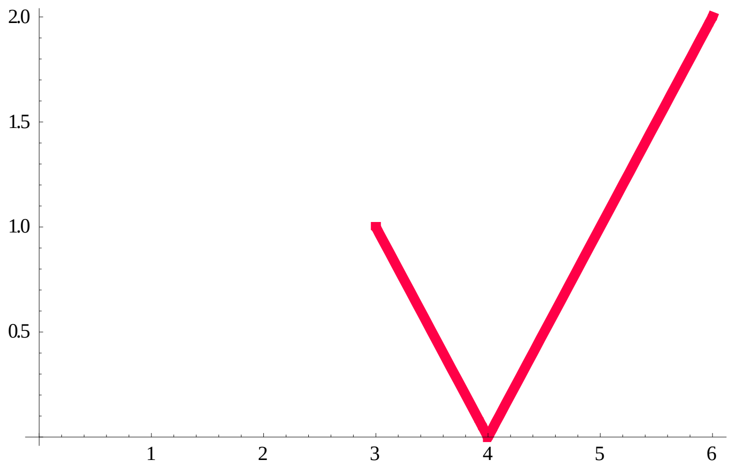
$$F'(x) = \frac{\ln x}{x^2}$$

3. *[5 pts]* Evaluate

$$\int_3^6 |\,4 - x\,|\; dx$$

Solution:

Since y = |4 – x| is piecewise linear, we can easily sketch its graph:



Note that the area under this curve over [3, 4] is simply the area of a triangle, ½ (1)(1) = ½ and the area under this cruve over [4, 6] is again the area of a triangle, ½ (2)(2) = 2. Thus the area under this curve over [3, 6] is ½ + 2 = 5/2.

4. *[5 pts]* Using the method of *judicious guessing* or *substitution*, evaluate

$$\int (\cos x)\,\sqrt{3 + 2\sin x}\,dx$$

Solution: Let our first guess be $g_1(x) = (3 + 2\sin x)^{3/2}$.
Now $dg_1/dx = (3/2)\,(3 + 2\sin x)^{1/2}\,(0 + 2(1/3\cos x)$*, we see how to modify* g_1 *to obtain the correct anti-derivative:*
$g_2(x) = (1/3)\,g_1(x) = (1/3)(3 + 2\sin x)^{3/2}$. *Hence*

$$\int (\cos x)\,\sqrt{3 + 2\sin x}\,dx = (1/3)\left(3 + 2\sin x\right)^{3/2} + C$$

5. *[5 pts]* Find:

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$$

Solution: Since both the numerator and denominator tend towards 0 as $x \to 0$, we may apply l'Hopital's rule:

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x \to 0} \frac{e^x - 1}{2x} = \lim_{x \to 0} \frac{e^x}{2} = \frac{1}{2}$$

Here we have applied l'Hopital's rule a second time since both $e^x - 1$ and $2x$ tend toward 0 as $x \to 0$.

6. *[5 pts]* Suppose $f(x)$ is a function with the following properties:

- $f(0) = 4$
- $f(2) = 0$
- $f(x)$ is decreasing on [0, 2]
- $f(x)$ is concave down on [0, 2]

Suppose that $g(x) = \int_0^x f(t)\, dt$.

a. Is $g(x)$ increasing or decreasing on [0, 2]? No explanation necessary.

Solution: Using the FTC, we find $dg/dt = f(t) > 0$.
Thus g is increasing on [0, 2].

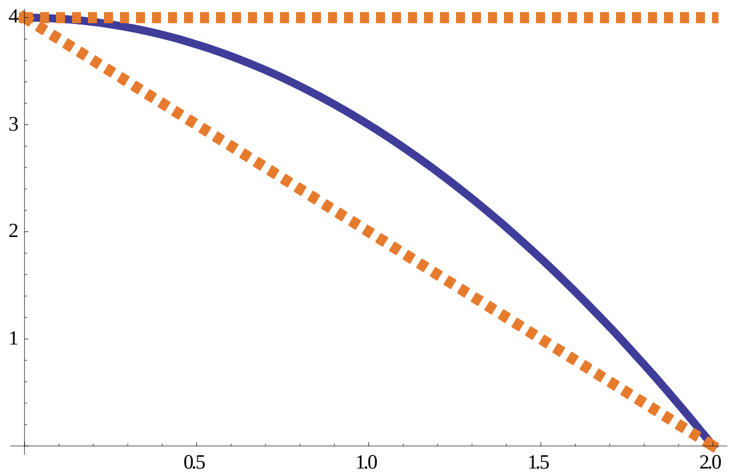
b. Is $g(x)$ concave up or concave down on [0, 2]? No explanation necessary.

Solution: Assuming that f is differentiable on (0, 2),
$d^2g/dt^2(g) = d/dt\,(dg/dt) = d/dt\,(f(t)) < 0$ since we are given that f is decreasing on [0, 2]. We conclude that g is concave down on [0, 2].

c. Sketch a graph of $f(x)$ which satisfies the above conditions and use it to explain why

$$4 < \int_0^2 f(x)\, dx < 8.$$

Solution:



Since the area beneath f(x) above [0, 2] must be strictly smaller than the area of the rectangle with height 4 and width 2, we obtain: $\int_0^2 f(x)\,dx < 8.$

Since f(x) lies above the line joining (0, 4) and (2, 0), the area beneath f(x) above [0, 2] must be strictly larger than the area of the right triangle with vertices (0, 0), (0, 4) and (2, 0). Since the area of this triangle is ½ (2)(4) = 4, we obtain:

$$4 < \int_0^2 f(x)\,dx.$$