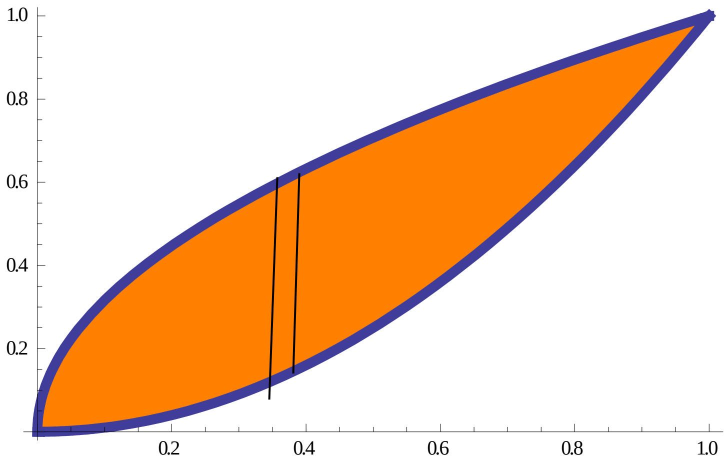
MATH 162 SOLUTIONS: QUIZ II

1. *[20 pts]* Consider the region R in the first quadrant that is bounded by the curves $y = x^2$ and $y = \sqrt{x}$. Sketch this region!



This region is rotated about the axis x = 2 to create a solid.

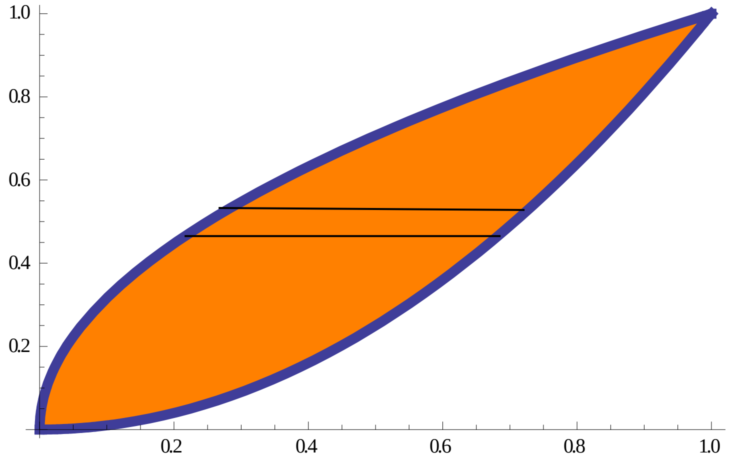
(a) Using the *shell method* write the volume of this solid of revolution as a Riemann integral. Show your work. Do not evaluate.

Solution: We choose a vertical rectangle by fixing x, where $0 \le x \le 1$. Rotating this rectangle about the axis x = 2, we obtain a shell of height $\sqrt{x} - x^2$ and of radius $2 - x$. Thus the shell will have area = $2\pi r h = 2\pi(2 - x)(\sqrt{x} - x^2)$. The thickness of the "label" (shell) is Δx. Thus the volume of the shell is $2\pi(2 - x)(\sqrt{x} - x^2)\,\Delta x$.

Summing, we obtain

$$Volume = 2\pi \int_0^1 (2 - x)(\sqrt{x} - x^2)\,dx$$

(b) Using the *washer method* write the volume of this solid of revolution as a Riemann integral. Show your work. Do not evaluate.



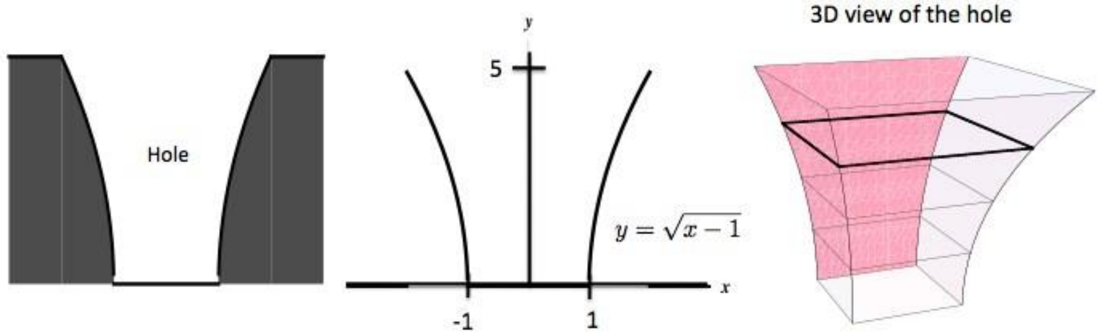
Solution: Here we need a horizontal rectangle since the washer method requires that the rectangle be perpendicular to the axis of rotation. To select a rectangle we specify y, where $0 \le y \le 1$. Now the outer radius of the washer is the horizontal distance from the upper curve to the axis $x = 2$. This is $2 - y^2$. Next, the inner radius of the washer is the horizontal distance from the lower curve to the axis $x = 2$. This distance is $2 - \sqrt{y}$. Now let the thickness of the washer be Δy.

Hence the volume of the washer is $(\pi r_{out}^2 - \pi r_{in}^2)\,\Delta y = (\pi(2-y^2)^2 - \pi(2-\sqrt{y})^2)\,\Delta y$

Summing these values yields

$$Volume = \pi \int_0^1 \left((2-y^2)^2 - \left(2 - \sqrt{y}\right)^2 \right) dy$$

2. *[13 points]* As part of an exploration assignment, a team of mining engineers dug a hole in the ground. The hole takes the shape of a solid region of known cross-section. The base region, which stands vertically, is pictured below. Cross-sections taken perpendicular to the *y*-axis are squares with one side lying on the *xy*-plane.



The variables *x* and *y* are given in meters.

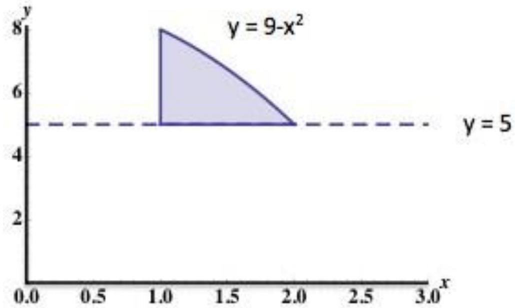
Take a slice of soil of thickness Δy meters located at *y* meters above bottom of the hole. Write a Riemann integral that represents the volume of the hole. Show all work to receive full credit.

Solution: Let y be a point on the y-axis between 0 and 5 that determines a slice of soil of thickness Δy that is y meters above the bottom of the hole. Then the area of the square cross-section is $(2y^2 + 2)^2$. Hence the volume of this square slice is:

$$\Delta V = F_{Slice}\,\Delta = (2(y^2+1))^2 \Delta y$$

$$\text{Thus volume} = 4\int_0^5 (y^2+1)^2 dy$$

3. *[20 points]* Consider the region in the xy-plane bounded by the curves $y = 9 - x^2$, $x = 1$, and $y = 5$. This region is pictured below.



Using the disk, washer or shell method (your choice!) write a Riemann integral that represents each of the following quantities. Do not evaluate.

a. The volume of the solid obtained by rotating the region about the y-axis.

Solution:

$$V = \int_5^8 \pi\left(\left(\sqrt{9-y}\right)^2 - 1\right) dy \quad \text{or} \quad V = \int_1^2 2\pi x\left(9 - x^2 - 5\right) dx = \int_1^2 2\pi x\left(4 - x^2\right) dx$$

b. The volume of the solid obtained by rotating the region about the line $y = 5$

Solution:

$$V = \int_1^2 \pi\left(4 - x^2\right)^2 dx \quad \text{or} \quad V = \int_5^8 2\pi(y-5)\left(\sqrt{9-y} - 1\right) dy$$

The book of nature is written in the language of mathematics.

- Galileo