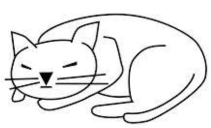
MATH 162 SOLUTIONS QUIZ III

1. *[10 pts]* Consider the parameterized curve: $x = e^t - 1$, $y = e^{2t}$, where $t \geq 0$, describing the position in the xy-plane of Mehitabel, the cat, at time *t*. (Note that Mehitabel is born at time t = 0 and is immortal.)

(a) By eliminating the parameter, *t*, express *y* as a function of *x*.

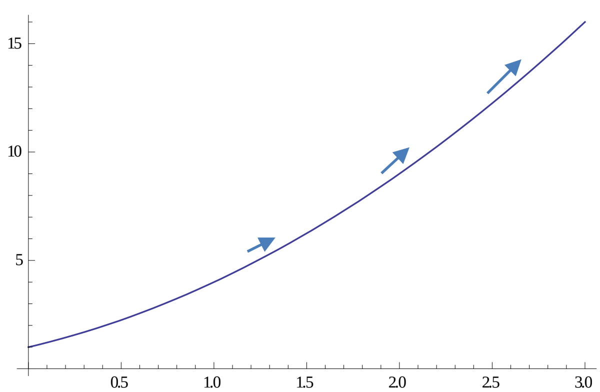
Solution: Since $x = e^t - 1$, $e^t = x + 1$. Hence $y = e^{2t} = (x + 1)^2$. Since Mehitabel is immortal, the domain of this function is $[0, \infty)$.

(b) *Sketch* the parameterized curve, using an arrow to indicate the direction of Mehitabel's journey. Also, indicate the birthplace of Mehitabel.



Solution: When t = 0 then x = 0 and y = 1. Thus Mehitabel is born at the point (0, 1) in the xy-plane.

As t > 0 increases, both x and y increase. Thus Mehitabel's direction of motion is northeast.



2. *[10 pts]* Express the arc length of the curve $y = x \ln x$ from x = 1 to x = 9 as a Riemann integral. *No need to evaluate the integral.*

Solution: Since $dy/dx = x(1/x) + \ln x = 1 + \ln x$, we have:

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+(1+\ln x)^2}\,dx$$

Finally:

$$s = \int_1^9 \sqrt{1+(1+\ln x)^2}\,dx$$

3. *[10 pts]* Express the *arc length* of the ellipse $\frac{x^2}{16}+\frac{y^2}{49}=1$ as a Riemann integral. *Do not try to evaluate this integral.*

(*Hint:* First parameterize the ellipse using *x(t) = 4 cos t* and *y(t) = 7 sin t* and then select an appropriate interval of *t* values.)

Solution: Following the hint, we choose the parameterization

x(t) = 4 cos t and *y(t) = 7 sin t* *where* $0 \le t \le 2\pi$.

Note that this domain $[0, 2\pi]$ *represents one complete rotation of the ellipse.*

Since dx/dt = -4 sin t and dy/dt = 7 cos t, we now use the arc-length formula:

$$ds = \sqrt{(dx/dt)^2 + (dy/dt)^2}\ dt = \sqrt{(-4\sin t)^2 + (7\cos t)^2}\ dx = \sqrt{16\sin^2 t + 49\cos^2 t}\ dt$$

Finally:

$$s = \int_0^{2\pi} \sqrt{16\sin^2 t + 49\cos^2 t}\ dt$$

4. *[10 pts]* Let *S* be the surface of revolution obtained by rotating the curve y = 1/x from x = 1 to x = 5 about the line x = -3.

Find a Riemann integral that expresses the surface area of this region. *(Do not evaluate the integral.)*

Solution: Recall that, symbolically, $dS = 2\pi\rho\, ds$.

Let's integrate with respect to x. Begin by noting that dy/dx = $-1/x^2$. SoP

$$ds = \sqrt{1+(dy/dx)^2}\ dx = \sqrt{1+(-1x^{-2})^2}\ dx = \sqrt{1+x^{-4}}\ dx$$

Next observe that for each x where $1 \le x \le 5$, we have p(x) = x – (-3) = x + 3.

Now:

$$ds = 2\pi(x+3)\ \sqrt{1+(-1x^{-2})^2}\ dx = (x+3)\sqrt{1+x^{-4}}\ dx$$

Finally:

$$S = \int_1^5 2\pi\rho(x)\,ds = \int_1^5 2\pi(x+3)\ \sqrt{1+x^{-4}}\ dx$$

5. [10 pts] Suppose that a cylindrical tank has height of 10 feet, base radius of 7 feet, and that the tank is half full of water. Find the amount of work necessary to move all of the water out of the top of the tank. (Assume that water weighs 64.5 lbs/ft^3.) *(Do not evaluate this integral.)*

Solution: Let $0 \le y \le 5$. Choose a horizontal slice passing through (0, y) having thickness Δy.
Then the volume of this slice is $\pi 7^2 \Delta y$.
The work performed in moving this slice 10 – y units from the x-axis is:

$\Delta W = \pi 7^2 \Delta y\ (10 - y)\ 64.5 = 49(64.5)\pi(10 - y)\ \Delta y$

Thus the total amount of work performed is given by

$$W = \int_0^5 49(64.5)\pi(10-y)\ dy \quad ft - lbs$$

I have hardly ever known a mathematician who was able to reason.

\- Stephen Hawking