MATH 162 SOLUTIONS: QUIZ VI

1. *[10 pts]* Find a formula for the general term a_n of each sequence below. Assume the pattern of the first few terms continues, and that the first term in each sequence corresponds to $n = 1$. For example, in part(a), $a_1 = 5$.

(a) 5, 8, 11, 14, 17, ...

This sequence grows linearly: that is, the next term is 3 greater than the current term.

$a_n = 5 + 3(n - 1)$

(b) $-3, 2, -\frac{4}{3}, \frac{8}{9}, -\frac{16}{27}, \ldots$

This sequence grows geometrically: that is, the next term is (-2/3) times the current term.

$a_n = 3(-1)^n(2/3)^{n-1}$

2. *[10 pts]* Let $a_n = \frac{2n}{3n+1}$

(a) Determine whether the sequence $\{a_n\}$ is convergent or not. If the sequence converges, find its limit. If the sequence diverges, explain why. *(Show your work.)*

Solution: We conjecture that $\{a_n\}$ converges to 2/3. Toward this end:

$$a_n = \frac{2n}{3n+1} = \frac{2}{3 + \frac{1}{n}} \rightarrow \frac{\mathbf{2}}{\mathbf{3 + 0}} = \frac{\mathbf{2}}{\mathbf{3}}$$

(b) Determine whether the series $\sum_{n=1}^{\infty} a_n$ is convergent or not. *Justify* your answer!

Solution: The series $\sum_{n=1}^{\infty} a_n$ diverges since the sequence of atoms does not converge to 0. (n^{th} term Test for Divergence)

3. *[28 pts]* For each of the following, determine whether the series is *convergent* or *divergent*. If the series *converges*, find the *sum*. If the series *diverges, explain why*.

(a) $\frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \frac{1}{12} + \frac{1}{15} + \cdots$

Solution: This series diverges since it equals 1/3 times the harmonic series:

Than is: $\frac{1}{3} + \frac{1}{6} + \frac{1}{9} + ... = \frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{n}$

(b) $4 + 3 + \frac{9}{4} + \frac{27}{16} + \ldots$

Solution: Since $a_{n+1} / a_n = 3/4$, *we recognize this series as geometric with* $R = 3/4$.

Since $|R| < 1$, *this series converges to:*

$$\frac{4}{1-3/4} = 16$$

(c) $\sum_{n=1}^{\infty} \frac{1}{4 + e^{-n}}$

Solution: This series diverges because of the n^{th} *term Test for Divergence.*

$$a_n = \frac{1}{4+e^{-n}} \to 1/4$$

(d) $\sum_{n=1}^{\infty} (\sqrt{2})^{-n}$

Solution: This series converges because it is geometric with common ratio of

$$\frac{\left(\sqrt{2}\right)^{-n-1}}{\left(\sqrt{2}\right)^{-n}} \to (\sqrt{2})^{-1} = \frac{1}{\sqrt{2}} < 1$$

The sum of this series is $$\frac{\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}=\frac{1}{\sqrt{2}\ -1}$$

4. *[30 pts]* Using an appropriate comparison test, determine whether each of the following series converges or diverges. *Explain your reasoning clearly.*

(a) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}-1}$

Solution: We know from the ("partial")p-test that $\displaystyle\sum_{1}^{\infty}\frac{1}{\sqrt{n}}$ diverges. Hence we will try to show that the given series diverges:

Toward this end: $\displaystyle\frac{1}{\sqrt{n}-1}>\frac{1}{\sqrt{n}}>0$

Finally, since $\displaystyle\sum_{1}^{\infty}\frac{1}{\sqrt{n}}$ diverges, the Comparison Test allows us to conclude that the original series diverges as well.

(b) $\displaystyle\sum_{n=1}^{\infty}(1+\cos^2 n)(2^{-n})$

Solution: Since $\displaystyle\sum_{1}^{\infty}2^{-n}$ *converges since it is geometric with common ratio of ½, we conjecture that our original series converges as well. Towards this end:*

$$0\le\frac{1+\cos^2 n}{2^n}\le\frac{1+1}{2^n}=\frac{1}{2^{n-1}}$$

Thus, since the geometric series $\displaystyle\sum_{1}^{\infty}\frac{1}{2^{n-1}}$ *converges, our original series must converge as well.*

(c) $\sum_{n=1}^{\infty} \frac{5^n + 3}{4^n + n + 7^n}$

Solution: $Let \ a_n = \frac{5^n}{4^n + n + 7^n}$

Note that the denominator is dominated by 7^n -------- ***not 11^n***

Since a_n is roughly $5^n / 7^n = (5/7)^n$, we conjecture that the original series converges.

Now: $0 \le a_n = \frac{5^n}{4^n + n + 7^n} < \frac{5^n}{7^n} = \left(\frac{5}{7}\right)^n$

Since $\sum_{n=1}^{\infty} \left(\frac{5}{7}\right)^n$ *is a convergent geometric series (with common ration of 5/7), we may invoke the comparison theorem to conclude that* $\sum_{n=1}^{\infty} \frac{5^n}{4^n + n + 7^n}$

5. The *factorial*, $n!$, of a positive integer n is defined to be $n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$. For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

Consider the series $\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$.

(a) *[10 pts]* Find the partial sums s_1, s_2, s_3, and s_4 of this series. *Write each of the four partial sums as a single fraction in lowest terms.*

Solution: Using the definition of partial sum, we have:

$s_1 = a_1 = 1/(2!) = ½$

$s_2 = s_1 + a_2 = ½ + 2/(3!) = ½ + 1/3 = 5/6$

$s_3 = s_2 + a_3 = 5/6 + 3/(4!) = 5/6 + 1/8 = 23/24$

$s_4 = s_3 + a_4 = 23/24 + a_4 = 23/24 + 4/(5!) = 23/24 + 1/30 = 119/120$

(b) *[5 pts]* Use your answers to part (a) to infer a general formula for s_n. (You do not have to verify your answer.)

Solution: We infer from the pattern above that

$$a_n = \frac{(n+1)!-1}{(n+1)!}$$

(c) *[4 pts]* Use your answer in part (b) to find the *sum of the series*.

Solution: $$a_n = \frac{(n+1)!-1}{(n+1)!} = 1 - \frac{1}{(n+1)!} \to 1 - 0 = 1$$

As a net is made up of a series of ties, so everything in this world is connected by a series of ties. If anyone thinks that the mesh of a net is an independent, isolated thing, he is mistaken. It is called a net because it is made up of a series of interconnected meshes, and each mesh has its place and responsibility in relation to the other meshes.

- Buddha