Recall that the Taylor Series for a function $f(x)$ centered at $x = a$ is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

The Taylor Series centered at $a = 0$ is called the Maclaurin Series and it has the form

$$f(0) + f'(a)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

The Maclaurin Series for some familiar functions are given below along with their radii of convergence.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \qquad R = \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots \qquad R = \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots \qquad R = \infty$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots \qquad R = 1$$

MATH 162– SOLUTIONS: QUIZ 8

For each of the following functions $f(x)$ and centers $x = a$, find the first four non-zero terms of the Taylor series and the radius of convergence.

(a) $f(x) = xe^{2x}$ centered at $x = 0$

Solution: $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$ converges for all x.

Replacing x by 2x in the above, we have

$e^{2x} = 1 + \frac{2x}{1!} + \frac{2^2x^2}{2!} + \frac{2^3x^3}{3!} + \frac{2^4x^4}{4!} + \ldots$ also converges for all x.

Finally, f(x) = x $e^{2x} = x + \frac{2x}{1!} + \frac{2^2x^2}{2!} + \frac{2^3x^3}{3!} + \ldots$

The first four non-zero terms of this series are:

$$p_4(x) = x + \frac{2x}{1!} + \frac{2^2x^2}{2!} + \frac{2^3x^3}{3!}$$

The series converges for all x.

(b) $f(x) = \frac{1}{1+4x}$ centered at $x = 0$

Solution: The geometric series $\frac{1}{1-r} = 1 + r + r^2 + r^3 + r^4 + \ldots$ converges for all r, 0<r<1.

Replacing r by –4x we obtain

$$\frac{1}{1+4x} = 1 + 4x + 4^2x^2 + 4^3x^3 + 4^4x^4 + \ldots$$

As we have replaced r by 4x, our new series will converge for |4x| < 1, that is: |x| < ¼

So the radius of convergence is R = ¼ .

2. Use the definition of the Taylor series to find the first five non-zero terms of the series for $f(x) = \ln x$ centered at $x = 1$

Solution: Computing the first 5 derivatives of f:

$f(x) = \ln x$

$f'(x) = 1/x$

$f''(x) = -1/x^2$

$f'''(x) = 1/x^3$

$f^{(4)}(x) = -1/x^4$

$f^{(5)}(x) = 1/x^5$

Replacing x by 1:

$f(1) = \ln 1 = 0$

$f'(1) = 1$

$f''(1) = -1$

$f'''(1) = 1$

$f^{(4)}(1) = -1$

$f^{(5)}(1) = 1$

Thus the first five non-zero terms are:

$$p_5(x) = \frac{1}{1!}(x-1) - \frac{1}{2!}(x-1)^2 + \frac{1}{3!}(x-1)^3 - \frac{1}{4!}(x-1)^4 + \frac{1}{5!}(x-1)^5$$

3. Find the limit using Taylor series. Do not use l'Hˆopital's Rule.

$$\lim_{x \to 0} \frac{e^{x^2} - 1 - x^2}{x^4}$$

Solution: $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$ converges for all x.

Substituting x^2 for x in the above, we have

$e^{x^2} = 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \ldots$ also converges for all x. Hence

$$\lim_{x\to 0} \frac{e^{x^2} - 1 - x^2}{x^4} = \lim_{x\to 0} \frac{\left(1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \ldots\right) - 1 - x^2}{x^4} = \lim_{x\to 0} \frac{\left(\frac{x^4}{2!}\right)}{x^4} = \frac{1}{2}$$

4. Find the exact value of the series

$$1 - \ln 2 + \frac{(\ln 2)^2}{2!} - \frac{(\ln 2)^3}{3!} + \frac{(\ln 2)^4}{4!} - \frac{(\ln 2)^5}{5!} + \cdots$$

Solution: Since $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$ replacing *x* by $-x$,

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots$$

Thus $1 - \frac{\ln 2}{1!} + \frac{(\ln 2)^2}{2!} - \frac{(\ln 2)^3}{3!} + \frac{(\ln 2)^4}{4!} - \ldots = e^{-\ln 2} = \left(e^{\ln 2}\right)^{-1} = 1/2$

5. Suppose that $\sum_{n=0}^{\infty} a_n x^n$ is a Maclaurin series for $f(x)$ and that the radius of convergence is infinite. Circle the letter which gives the value of $f'(1)$. No explanation necessary.

(a) 0

(b) a_1

(c) $\sum_{n=0}^{\infty} a_n$

(d) $\sum_{n=1}^{\infty} na_n$

(e) $\sum_{n=1}^{\infty} na_n^{n-1}$

Solution:

$$f'(x) = \frac{d}{dx}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{d}{dx} a_n x^n = \sum_{n=1}^{\infty} na_n x^{n-1}$$

Thus $f'(1) = \sum_{n=1}^{\infty} na_n$ and so the correct choice is (e).

6. The Taylor series for $f(x)$ centered at $x = 1$ is given by

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n}(x-1)^n.$$

a. Find the first four non-zero terms of the Taylor series for $f'(x)$ centered at $x = 1$.

Solution: We can write out the first several terms, or differentiate the general term.

$$\frac{d}{dx}\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n}(x-1)^n = \sum_{n=0}^{\infty} \frac{d}{dx}(-1)^{n+1} \frac{2^n}{n}(x-1)^n = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{2^n}{n}\frac{d}{dx}(x-1)^n =$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n} n(x-1)^{n-1} = \sum_{n=1}^{\infty} (-1)^{n+1} 2^n (x-1)^{n-1}$$

Writing the first four non-zero terms of f '(x) centered about x = 1:

$$p_4(x) = \sum_{n=1}^{4} (-1)^{n+1} 2^n \ (x-1)^{n-1} = 2 - 2^2(x-1) + 2^3(x-1)^2 - 2^4(x-1)^3$$

b. The Taylor series for $f'(x)$ you found in part (a) is a geometric series. What is the common ratio of this geometric series?

Solution: The ratio between successive terms is: -2(x – 1)