Math 116 — Final Exam

December 15, 2011

Name: ____

Instructor: ______________________________ Section: ______________

1. **Do not open this exam until you are told to do so.**
2. This exam has 12 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.

Problem	Points	Score
1	10	
2	12	
3	10	
4	9	
5	10	
6	7	
7	12	
8	12	
9	11	
10	7	
Total	100	

You may find the following expressions useful. And you may not. But you may use them if they prove useful.

"Known" Taylor series (all around $x = 0$):

$$\sin(x) = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} + \cdots\cdots \qquad \text{for all values of } x$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n\, x^{2n}}{(2n)!} + \cdots \qquad \text{for all values of } x$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{for all values of } x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n+1}x^n}{n} + \cdots \qquad \text{for } -1 < x < 1$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}\, x^2 + \frac{p(p-1)(p-2)}{3!}\, x^3 + \cdots \qquad \text{for } -1 < x < 1$$

1. [10 points] Indicate if each of the following is true or false by circling the correct answer. No justification is required.

a. [2 points] Let a_n be a sequence of positive numbers. If $a_n \leq \frac{7^n}{2^{3n-1}}$ for all values of $n \geq 1$, then a_n must converge.

True False

b. [2 points] The trapezoid rule is guaranteed to give an underestimate of $\int_{-\pi}^{\pi} \cos t dt$.

True False

c. [2 points] If the area A under the graph of a positive continuous function $f(x)$ is infinite, then the volume of the solid generated by rotating A around the x-axis could be either infinite or finite depending on the function $f(x)$.

True False

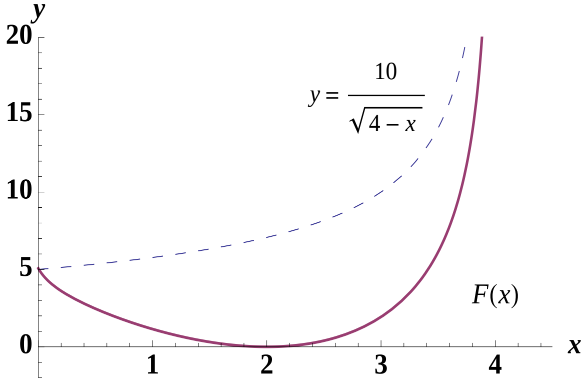
d. [2 points] If $H(x) = \int_0^x f(t)g(t)dt$, then $H'(x) = f'(x)g(x) + f(x)g'(x)$.

True False

e. [2 points] If $(x(t), y(t))$ gives a parametrization of the unit circle centered at the origin, then $\int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = 2\pi$.

True False

2. [12 points] The graph of $F(x)$ is given below. The function $F(x)$ is defined for $0 \leq x < 4$, and its graph is given below. As shown $F(x)$ has a vertical asymptote at $x = 4$. Let $G(x)$ be the antiderivative of $F(x)$ with $G(1) = 1$.



a. [2 points] For what values of x is $G(x)$ increasing?

b. [2 points] For what values of x is $G(x)$ concave up?

c. [2 points] Find a formula for $G(x)$ in terms of $F(x)$.

d. [4 points] Is $\int_0^4 \frac{10}{\sqrt{4-x}} dx$ convergent or divergent? If it is convergent, find its exact value.

e. [2 points] Does $\lim_{x \to 4^-} G(x)$ exist? Justify.

3. [10 points] In order to fuel a late-night study session at the UGLi, you pull out a can of Bolt Kola, a highly caffeinated soft drink. This particular brand of pop comes in a cylindrical, aluminum can with a removable top. You want to know how much force is exerted on the sides of the can by the drink. You pull out your trusty ruler and find the base of the can has a radius of 3.5 centimeters (0.035 meters) and a height of 16 centimeters (0.16 meters). A quick Google search informs you that the density of Bolt Kola is 1030 kg/m^3.

a. [2 points] Calculate the force exerted by the drink on the bottom of the can.

b. [5 points] Write an expression giving the force exerted by the drink on a slice of the cylindrical wall of the can h meters above the base and of thickness Δh.

c. [3 points] Calculate the total force exerted by the drink on the sides of the can (with the top removed). Show all work to receive full credit.

4. [9 points] Ramon starts depositing \$10,000 each year at his 25th birthday into a retirement account and continues until his 45th birthday. After this point, he does not touch the account until he is 65. The retirement account accrues interest at a rate of 3% compounded annually.

a. [3 points] Let R_n be the amount of money *in thousands* of dollars in Ramon's retirement account after n years from his initial deposit. Find an expression for R_0, R_1 and R_2.

b. [3 points] Find a closed form expression (an expression that does not involve a long summation) for how much money Ramon has in his retirement account at his 45th birthday.

c. [3 points] Find a closed form expression for how much money Ramon has in his retirement account when he is 65 years old. Compute its value.

5. [10 points] When a voltage V in volts is applied to a series circuit consisting of a resistor with resistance R in ohms and an inductor with inductance L, the current $I(t)$ at time t is given by

$$I(t) = \frac{V}{R}\left(1 - e^{-\frac{Rt}{L}}\right) \qquad \text{where } V, R, \text{ and } L \text{ are constants.}$$

a. [2 points] Show that $I(t)$ satisfies

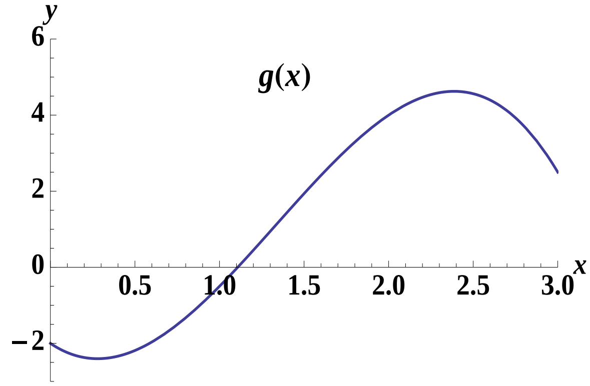
$$\frac{dI}{dt} = \frac{V}{L}\left(1 - \frac{R}{V}I\right).$$

b. [6 points] Find a Taylor series for $I(t)$ about $t = 0$. Write the first three nonzero terms and a general term of the Taylor series.

c. [2 points] Use the Taylor series to compute

$$\lim_{t \to 0} \frac{I(t)}{t}.$$

6. [7 points] Let $f(x) = \frac{1}{3x-2}$ and $g(x)$ be the function whose graph is shown below.



a. [3 points] Let $P_2(x) = a + b(x-2) + c(x-2)^2$ be the second order Taylor polynomial approximating $g(x)$ for x near 2. What can you say about the signs of the coefficients a, b and c?

b. [4 points] Find the second order Taylor polynomial approximating $f(x)$ for x near -1.

7. [12 points] Determine whether the following series converge or diverge (circle your answer). Be sure to mention which tests you used to justify your answer. If you use the comparison test or limit comparison test, write an appropriate comparison function.

a. [3 points] $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{1+2\sqrt{n}}$

b. [4 points] $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

c. [5 points] $\displaystyle\sum_{n=1}^{\infty}\frac{\cos(n^2)}{n^2}$

8. [12 points] Let

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n(n+1)}(x+1)^{2n}$$

a. [3 points] At $x = -3$, does the series converge absolutely, conditionally or diverge?

b. [2 points] Using just your answer in (a), state the possible values for the radius of convergence R could be. Justify.

c. [7 points] Find the interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n(n+1)}(x+1)^{2n}$$

9. [11 points] Philip J. Frye has a bank account with Big Apple Bank that compounds interest at a continuous annual rate of 1%. His account has a balance of \$300 at midnight of January 1, 2000, when Frye is cryogenically frozen for 1000 years. The entire time he is frozen, his account accumulates interest. Include units in your answers where appropriate.

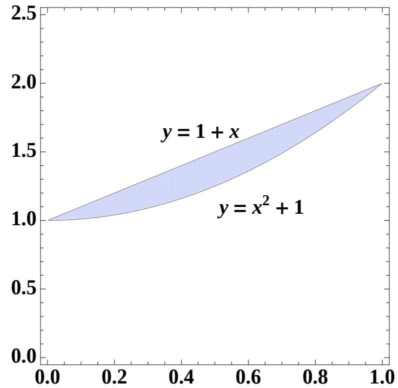
a. [2 points] Write a differential equation that models Frye's account balance $M(t)$, where M is given in dollars and t is the number of years since January 1, 2000. List any initial conditions.

b. [4 points] Solve your differential equation from (a) to find the balance in Frye's account after he is awoken in the year 3000. Show all your work.

c. [2 points] Suppose that Big Apple Bank charges an annual fee of \$5 to maintain the account, withdrawn continuously over the course of the year. Write a new differential equation for $M(t)$, the balance in Frye's bank account.

d. [3 points] How large must the initial deposit in Frye's account be at Big Apple Bank in order for the account to be profitable for him? Justify your answer mathematically.

10. [7 points] A metal thin plate has density $\delta(x) = 1 + x$ kg per square meter. The shape of the plate is bounded by the curves $y = 1 + x$ and $y = 1 + x^2$ for $0 \leq x \leq 1$ as shown below.



a. [3 points] Find the exact value of the mass of the plate. Show all your work.

b. [4 points] Let $(\bar{x}, \bar{y})$ be its center of mass. Write a formula for $\bar{x}$ and evaluate your formula to find the exact value of $\overline{x}$. Show all your work.