WORKSHEET XVI

OPERATIONS ON POWER SERIES

1. (a) Consider the power series $f(x) = 1/(1 – x) = 1 + x + x^2 + x^3 + \ldots$ What is its interval of convergence?

   (b) Find the series for $f'(x)$ *and* $f''(x)$.

2. Using the power series obtained in 1(b) for $f'(x)$, determine the sum of the series $\sum_{n=1}^{\infty} \frac{n}{3^n}$

3. The series

   $\sin x = x – x^3/3! + x^5/5! – x^7/7! + x^9/9! – x^{11}/11! + \ldots$

   converges to sin x for all real x.

   (a) Find the first six terms of a series for cos x. For which values should the series converge?

   (b) By replacing x by 2x in the series for sin x, find a series that converges to sin 2x for all *x*.

   (c) Using series multiplication, find a series that converges to 2 sin x cos x.

4. (a) On which interval does the series $1/(1 + t) = 1 – t + t^2 – t^3 + t^4 – t^5 +\ldots$ converge?

   (b) Integrating both sides over the interval [0, x], find the first six terms of a series that converges to ln(1 + x).

5. Find a power series representation of $1/(1 + x^2)$.
6. Find a power series representation of $1/(2 + x^2)$.
7. Find a power series representation of $x^4/(2 + x^2)$.
8. Find a power series representation of $1/(1 – x)^2$. Differentiate a well-known geometric sum.

9. The series $e^x = 1 + x + x^2/2! + x^3/3! + x^4/4! + x^5 /5! + \ldots$ converges to e^x for all real *x*.

   (a) Find a series for $(d/dx)\, e^x$.

(b) Find a series for $\int e^x \, dx$

(c) In the series for e^x, replace x by –x to find a series expansion of e^{-x}.

(d) In (c), replace x by x^2 to find a series expansion of e^{-x^2}.

10. (a) Beginning with the series for $1/(1 + x^2)$, find a series expansion of arctan x.

(b) Find a series expansion for $\int \frac{\arctan x}{x} \, dx$.

11. Using a series representation for sin 3x, find values of *r* and *s* for which

$$\lim_{x \to 0} \left(\frac{\sin 3x}{x^3} + \frac{r}{x^2} + s \right) = 0$$

12. *[Stewart]* Using the power series for arctan x, prove that the following series converges to π:

$$\pi = 2\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}$$

I used to love mathematics for its own sake, and I still do, because it allows for no hypocrisy and no vagueness….

- Stendhal, The Life of Henri Brulard