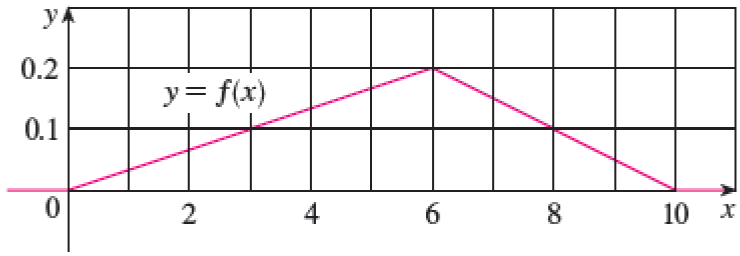
1. Consider the following function whose graph is shown.



(a) Let X be a random variable with probability density function represented by this graph. Explain why in fact this is the graph of a probability density function.

Solution: Area under y = f(x) is ½ (10)(0.2) = 1. Also f(x) is non-negative and defined on the real line.

(b) Find the probability that X < 3. (Express your answer to the nearest hundredth.)

Solution: $P(X < 2) = \int_{-\infty}^{3} f(x)dx = \int_{0}^{3} f(x)dx$ = ½ (3)(0.1) = 0.15

(c) Find the probability that 3 < X <8. (Express your answer to the nearest hundredth.)

Solution:

$$P(3 < X < 8) = \int_{3}^{8} f(x)dx = \textit{area beneath the curve } y = f(x) \textit{ over } [3,8] = 5\,(0.1) + \frac{1}{2}(5)(0.1) = 0.75$$

2. For each of the following improper integrals of the second kind, determine convergence or divergence. In the case of convergence, *compute the exact value of the integral. Simplify when possible.*

$$(A) \quad \int_{0+}^{1} \frac{11+x^2}{x^3}\, dx$$

Solution:

Using the Comparison Test:

We see that $\frac{11+x^2}{x^3} \geq \frac{x^2}{x^3} = \frac{1}{x} > 0$

$$\textit{Since } \int_{0+}^{1} \frac{1}{x}\, dx \textit{ diverges by the } p-\textit{test},$$

$$\textit{it follows that } \int_{0+}^{1} \frac{11+x^2}{x^3}\, dx \textit{ diverges as well.}$$

$$(B) \quad \int_0^{1-} \frac{1}{\sqrt{1-x^2}}\,dx$$

Solution: $\int_{0+}^{1} \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{p\to 0+} \int_p^1 \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{p\to 0+} \arcsin x \;\Big|_p^1 =$

$$\lim_{p\to 0+} (\arcsin 1 - \arcsin p) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Hence the given improper integral converges.

$$(C) \quad \int_0^{\frac{\pi}{2}-} \tan x\; dx$$

Solution: This integral diverges because:

$$\int_0^{\frac{\pi}{2}-} \tan x\; dx = \lim_{c\to\frac{\pi}{2}-} \int_0^c \tan x\; dx = \lim_{c\to\frac{\pi}{2}-} \frac{1}{2}(-\ln|\cos x|)\Big|_0^c =$$

$$-\frac{1}{2}\lim_{c\to\frac{\pi}{2}-}\big(\ln|\cos c| - \ln(\cos 0)\big) = -\frac{1}{2}\lim_{c\to\frac{\pi}{2}-}\big(\ln|\cos c|\big) \to \infty$$

since $\cos(c) \to 0+$ *as* $c \to (\pi/2)-$.

$$(D) \quad \int_{0+}^{1} \ln\left(\frac{1}{x}\right) dx$$

Solution: First note that an anti $-$ *derivative of* $\ln x$ *is* $x\ln x - x$.

Hence

$$\int_{0+}^{1} \ln\left(\frac{1}{x}\right) dx = -\lim_{p\to 0+} \int_{p}^{1} \ln x \; dx = -\lim_{p\to 0+} \left(x\, lnx - x)\Big|_{p}^{1} \right) =$$

$$-\lim_{p\to 0+} \big((0-1) - (p\, ln\, p - p)\big) = 1 - 0 = 1$$

Hence the given improper integral converges.

3. The life span (in years) of a vampire bat can be modeled by a random variable with probability density function

$$f(x) = \begin{cases} ce^{-x/10} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the constant c. Show all work. *[Hint: All bats must die eventually.]*

Solution:

$$1 = \int_{0}^{\infty} ce^{-x/10}\; dx = c \lim_{c\to\infty} \int_{0}^{c} e^{-x/10}\; dx = c \lim_{c\to\infty} \left(-10e^{-x/10} \Big|_{0}^{c} \right) =$$

$$c \lim_{c\to\infty} -10\left(e^{-c/10} - e^{0}\right) = 10c$$

Hence c = 1/10

(b) Find the probability that a randomly chosen bat will live *longer than* 11 years. Express your answer to the nearest hundredth.

Solution:

$$P(bat\ lives\ longer\ than\ 11\ years) = \frac{1}{10}\int_{11}^{\infty} e^{-x/10}\ dx =$$

$$\frac{1}{10}\lim_{c\to\infty}\int_{11}^{c} e^{-x/10}\ dx = \frac{1}{10}\lim_{c\to\infty}\left((-10)e^{-x/10}\Big|_{11}^{c}\right) =$$

$$(-1)\lim_{c\to\infty}\left(e^{-c/10} - e^{-11/10}\right) = e^{-11/10} \approx 0.33$$