MATH 162 SOLUTIONS: QUIZ IV

23 FEBRUARY 2018

1. Find a formula for the general term a_n of the sequence given below, assuming that the pattern of the first few terms continues. (Assume that n begins with 1.)

   {3, 8, 13, 18, 23, . . .}

   Answer: $a_n = 5n - 2$

2. *Consider the sequence* $\{a_n\}$ *defined by* $a_n = \frac{3^n}{1+\pi^n}$

   Determine whether the sequence converges or diverges. If it converges, find the limit. (If the sequence diverges, write DNE.)

Solution: $a_n = \frac{3^n}{1+\pi^n} \approx \frac{3^n}{\pi^n} = \left(\frac{3}{\pi}\right)^n \to 0$

3. *Consider the sequence* $\{a_n\}$ *defined by* $a_n = \sqrt[n]{4^{3+4n}}$

   Determine whether the sequence converges or diverges. If it converges, find the limit. (If the sequence diverges, write DNE.)

Solution: $a_n = \sqrt[n]{4^{3+4n}} = 4^{\frac{3+4n}{n}} = 4^4\, 4^{\frac{3}{n}} \to 4^4 = 256.$

Thus the sequence converges to 256.

4. *Consider the sequence* $\{a_n\}$ *defined by* $a_n = n \sin\frac{3}{n}$ Determine whether the sequence converges or diverges. If it converges, find the limit. (If the sequence diverges, write DNE.)

Solution: $a_n = n\, sin\frac{3}{n} = 3\,\frac{\sin\frac{3}{n}}{\frac{3}{n}} \to 3$

5. *Consider the sequence* $\{a_n\}$ *defined by* $a_n = \ln(5n^2 + 7) - \ln(n^2 + 3)$

   Determine whether the sequence converges or diverges. If it converges, find the limit. (If the sequence diverges, write DNE.)

Solution: $a_n = \ln(5n^2 + 7) - \ln(n^2 + 3) = \ln\frac{5n^2+7}{n^2+3} \to \ln 5$

Thus a_n *converges to ln 5.*

6. *Consider the sequence* $\{a_n\}$ *defined by the pattern* {0, 5, 0, 0, 5, 0, 0, 0, 5, ...}

   Determine whether the sequence converges or diverges. If it converges, find the limit. (If the sequence diverges, write DNE.)

Solution: This sequence fails to converge because the sequence continues to oscillate between 0 and 5.

7. *Consider the sequence* $\{a_n\}$ *defined by* $a_n = \sqrt{4^n + 5^n}$. Determine whether the sequence converges or diverges. If it converges, find the limit. (If the sequence diverges, write DNE.)

Solution:

$$a_n = \sqrt{4^n + 5^n} > \sqrt{4^n} = 4^{\frac{n}{2}} \to \infty.$$ Hence the sequence $\{a_n\}$ diverges.

8. *Consider the sequence $\{a_n\}$ defined by* $a_n = \frac{e^n + e^{-n}}{e^{2n} - 1}$.
Determine whether the sequence converges or diverges.

Solution:

$$0 < a_n = \frac{e^n + e^{-n}}{e^{2n} - 1} < \frac{e^n + e^n}{e^{2n} - \frac{1}{2}e^{2n}} = \frac{4}{e^n} \to 0$$

Hence, by the Squeeze Theorem,

$$a_n \to 0$$

9. Determine whether the series $\sum_{n=1}^{\infty}(\arctan(n^2))$ converges or diverges. If the series converges, find the sum. (If the series diverges, write DNE. $\sum_{n=1}^{\infty} \arctan(n^2)$)

Solution: Since $\arctan(n^2) \to \frac{\pi}{4} \neq 0$, *the n^th term test says that our series* $\sum_{n=1}^{\infty}(\arctan(n^2))$ *is divergent.*

10. Determine whether the geometric series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{(-6)^{n-1}}{7^n}$$

If it is convergent, find its sum. (If the quantity diverges, enter DIVERGES.)

Solution: This series is geometric, with first term of 1/7 and common ratio of -6/7. Since |-6/7| < 1, this geometric series converges.

Its sum is $\frac{a}{1-r} = \frac{\frac{1}{7}}{1-(-\frac{6}{7})} = \frac{1}{13}.$

11. Determine whether the series $\sum_{n=1}^{\infty} \ln\left(\frac{n^2+9}{8n^2+9}\right)$ is convergent or divergent.

If it is convergent, find its sum. (If the quantity diverges, enter DIVERGES.)

Solution: Since $\ln\left(\frac{n^2+9}{8n^2+9}\right) \to \ln\frac{1}{8} \neq 0$, *the n^th term test may be applied. So we conclude that our series* $\sum_{n=1}^{\infty}(\arctan(n^2))$ *is divergent.*

12. Determine whether the series $\sum_{n=1}^{\infty}\left(e^{\frac{5}{n}} - e^{\frac{5}{n+1}}\right)$ is convergent or divergent.

Solution: This series is telescoping.

$$s_j = \sum_{n=1}^{j} \left(e^{\frac{5}{n}} - e^{\frac{5}{n+1}}\right) = \left(e^{\frac{5}{1}} - e^{\frac{5}{2}}\right) + \left(e^{\frac{5}{3}} - e^{\frac{5}{4}}\right) + \ldots + \left(e^{\frac{5}{j}} - e^{\frac{5}{j+1}}\right) =$$

$$e^5 - e^{\frac{5}{j+1}} \rightarrow e^5 - 1$$

Thus $\sum_{n=1}^{\infty} \left(e^{\frac{5}{n}} - e^{\frac{5}{n+1}}\right)$ is convergent.

EXTRA CREDIT

The **Kolakoski Sequence** begins

1,2,2,1,1,2,1,2,2,1,2,2,1,1,2,1,1,...

The sequence contains only the numbers 1 and 2. They appear in either a run of one or a run of two. If we start at the beginning of the sequence, then, as illustrated below, the length of the runs (written below the sequence) recreates the original sequence.

1,2,2,1,1,2,1,2,2,1,2,2,1,1,2,1,1

1 2 2 1 1 2 1 2 2 1 2

Find the next 10 terms of the sequence.

Solution:

2, 2, 1, 2, 1, 1, 2, 1, 2, 2

1,2,2,1,1,2,1,2,2,1,2,2,1,1,2,1,1,2,2,1,2,1,1,2,1,2,2,1,1,2,1,1,2,1,2,2,1,2,2,1,1,2,1,2,2,...

It is an unsolved problem whether the density of 1s to 2s for the Kolakoski sequence is 1/2. In other words, if $O(n)$ is the number of 1s appearing among the first n terms and $T(n)$ is the number of 2s appearing among the first n terms, then it is conjectured that $\lim_{n\to\infty} \frac{O(n)}{T(n)} = 0.5$, but nobody knows how to prove it. In 1993, Vaclav Chvatal, a computer scientist and software engineer at Concordia University in Canada has shown that the density of 1s is less than 0.50084.