MATH 162 SOLUTIONS: QUIZ VI 23RD MARCH 2018

Part I *(4 pts)*

Suppose that $\sum_{n=0}^{\infty} a_n x^n$ is a Maclaurin series for $f(x)$ and that the radius of convergence is infinite. Circle the letter which gives the value of $f'(1)$. No explanation necessary.

(a) 0

(b) a_1

(c) $\sum_{n=0}^{\infty} a_n$

(d) $\sum_{n=1}^{\infty} n a_n$

(e) $\sum_{n=1}^{\infty} n a_n^{n-1}$

Solution:

$$f'(x) = \frac{d}{dx}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{d}{dx} a_n x^n = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Thus $f'(1) = \sum_{n=1}^{\infty} n a_n$ and so the correct choice is (d).

Part II *(10 pts each)*

Answer any 4 of the following 5 problems. You may answer all 5 to obtain extra credit.

1. Let $G(x) = e^{2x}$. Using an appropriate Maclaurin series, compute $G^{(2018)}(0)$. *(Do not try to simplify your answer.)*

Solution:

$$e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \ldots + \frac{t^n}{n!} + \ldots$$

$$\text{So } e^{2x} = 1 + \frac{2x}{1!} + \frac{2^2 x^2}{2!} + \frac{2^3 x^3}{3!} + \frac{2^4 x^4}{4!} + \frac{2^5 x^5}{5!} + \ldots + \frac{2^n x^n}{n!} + \ldots$$

Here the coefficient of x^{2018} is $2^{2018}/ 2018!$

Now the general Maclaurin series has, for its x^{2018} coefficient:

$$\frac{G^{(2018)}(0)}{2018!}$$

Hence $\frac{G^{(2018)}(0)}{2018!} = \frac{2^{2018}}{2018!}$

Finally, $G^{(2018)}(0) = 2^{2018}$

2. Suppose that the Maclaurin series of $g(x) = \sinh^{-1}(x)$ is given by:

$$\sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1}$$

Find $g^{(5)}(0)$. *Do not simplify!*

Solution: The coefficient of x^5 corresponds to n = 2. This coefficient is

$$\frac{(-1)^2 (4)!}{4^2 (2!)^2 5} = \frac{4!}{4^2 (2!)^2 5} = \frac{3!}{4^2 5} = \frac{3}{40}$$

Now the general Maclaurin series has, for its x^5 coefficient:

$$\frac{g^{(5)}(0)}{5!}$$

Hence $\frac{g^{(5)}(0)}{5!} = \frac{3}{40}$

Finally $g^{(5)}(0) = \frac{(3)(5!)}{40} = 9$

3. Find the first four non-zero terms of the Maclaurin series expansion of:

$$h(x) = x^4 e^{x^5}$$ *(No need to simplify!)*

Solution:

$$e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \ldots + \frac{t^n}{n!} + \ldots$$

$$\text{So } e^{x^5} = 1 + \frac{x^5}{1!} + \frac{(x^5)^2}{2!} + \frac{(x^5)^3}{3!} + \frac{(x^5)^4}{4!} + \frac{(x^5)^5}{5!} + \ldots + \frac{(x^5)^n}{n!} + \ldots$$

$$= 1 + \frac{x^5}{1!} + \frac{x^{10}}{2!} + \frac{x^{15}}{3!} + \frac{x^{20}}{4!} + \frac{x^{25}}{5!} + \ldots + \frac{x^{5n}}{n!} + \ldots$$

Thus:

$$x^4 e^{x^5} = x^4 + \frac{x^9}{1!} + \frac{x^{14}}{2!} + \frac{x^{19}}{3!} + \frac{x^{24}}{4!} + \frac{x^{29}}{5!} + \ldots + \frac{x^{5n+4}}{n!} + \ldots$$

So the first four non-zero terms of the series expansion for h(x) are:

$$x^4 + \frac{x^9}{1!} + \frac{x^{14}}{2!} + \frac{x^{19}}{3!}$$

4. Find the *first four non-zero* terms of the Maclaurin series of g(x) = 3 sin x + cos(2x).

Solution:

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

So

$$3 \sin x = \frac{3x}{1!} - \frac{3x^3}{3!} + \frac{3x^5}{5!} + \cdots + (-1)^n \frac{3x^{2n+1}}{(2n+1)!} + \cdots$$

Next:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)\,!} + \ldots$$

$$So\ \ \cos 2x = 1 - \frac{2^2 x^2}{2!} + \frac{2^4 x^4}{4!} - \frac{2^6 x^6}{6!} + \frac{2^8 x^8}{8!} - \ldots + (-1)^n \frac{2^{2n} x^{2n}}{(2n)\,!} + \ldots$$

Finally: g(x) = 3 sin x + cos(2x) = $\left\{\frac{3x}{1!} - \frac{3x^3}{3!} + \frac{3x^5}{5!} + \cdots + (-1)^n \frac{3x^{2n+1}}{(2n+1)!} + \cdots\right\} +$

$$\left(1 - \frac{2^2 x^2}{2!} + \frac{2^4 x^4}{4!} - \frac{2^6 x^6}{6!} + - \ldots + (-1)^n \frac{2^{2n} x^{2n}}{(2n)\,!} + \ldots\right)$$

$$= 1 + \frac{3x}{1!} - \frac{2^2 x^2}{2!} - \frac{3x^3}{3!} + \ldots$$

Thus the first four non-zero terms are:

$$1 + \frac{3x}{1!} - \frac{2^2 x^2}{2!} - \frac{3x^3}{3!}$$

5. Find the limit using Taylor series. Do not use l'Hôpital's Rule.

$$\lim_{x \to 0} \frac{e^{x^2} - 1 - x^2}{x^4}$$

Solution:

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$ converges for all x.

Substituting x^2 for x in the above, we have

$e^{x^2} = 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \ldots$ also converges for all x. Hence

$$\lim_{x \to 0} \frac{e^{x^2} - 1 - x^2}{x^4} = \lim_{x \to 0} \frac{\left(1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \ldots\right) - 1 - x^2}{x^4} = \lim_{x \to 0} \frac{\left(\frac{x^4}{2!}\right)}{x^4} = \frac{1}{2}$$

Extra Credit:

Find the exact value of the series

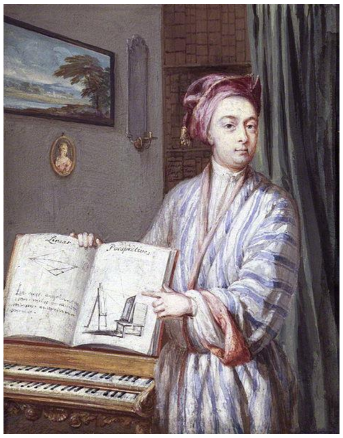
$$1 - \ln 2 + \frac{(\ln 2)^2}{2!} - \frac{(\ln 2)^3}{3!} + \frac{(\ln 2)^4}{4!} - \frac{(\ln 2)^5}{5!} + \cdots$$

Solution: Since $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$, replacing x by $-x$,

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots$$

Thus $1 - \frac{\ln 2}{1!} + \frac{(\ln 2)^2}{2!} - \frac{(\ln 2)^3}{3!} + \frac{(\ln 2)^4}{4!} - \ldots = e^{-\ln 2} = \left(e^{\ln 2}\right)^{-1} = 1/2$

Taylor was one of the few English mathematicians who could hold their own in disputes with Continental rivals, although even so he did not always prevail. Bernoulli pointed out that an integration problem issued by Taylor as a challenge to “non-English mathematicians” had already been completed by Leibniz in *Acta eruditorum*. Their debates in the journals occasionally included rather heated phrases and, at one time, a wager of fifty guineas. When Bernoulli suggested in a private letter that they couch their debate in more gentlemanly terms, Taylor replied that he meant to sound sharp and “to show an indignation”.



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