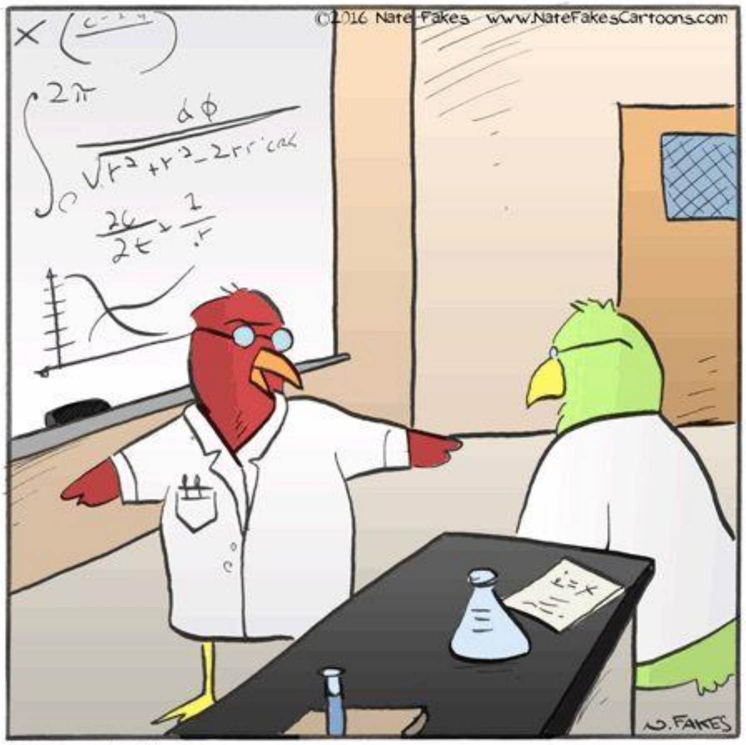
MATH 162 SOLUTIONS: QUIZ IX



"Well, I can't figure it out either, Petey. I wish we weren't bird brains!"

1. Express the *arc length* of the curve $y = x \ln x$ from x = 1 to x = 9 as a Riemann integral. *No need to evaluate the integral.*

Solution: Since dy/dx = x(1/x) + ln x = 1 + ln x, we have:

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+(1+\ln x)^2}\,dx$$

Finally:

$$s = \int_1^9 \sqrt{1+(1+\ln x)^2}\,dx$$

2. Let S be the surface of revolution obtained by rotating the curve y = 1/x from x = 1 to x = 5 about the line x = -3. Find a Riemann integral that expresses the surface area of this region. *(Do not evaluate the integral.)*

Solution: Recall that, symbolically, dS = 2$\pi\rho$ ds.

Let's integrate with respect to x. Begin by noting that dy/dx = -1/x². So

$$ds = \sqrt{1+(dy/dx)^2}\ dx = \sqrt{1+(-1x^{-2})^2}\ dx = \sqrt{1+x^{-4}}\ dx$$

Next observe that for each x where 1 $\le$x$\le$5, we have p(x) = x – (-3) = x + 3.

Now:

$$ds = 2\pi(x+3)\ \sqrt{1+(-1x^{-2})^2}\ dx = (x+3)\sqrt{1+x^{-4}}\ dx$$

Finally:

$$S = \int_1^5 2\pi\rho(x)\,ds = \int_1^5 2\pi(x+3)\,\sqrt{1+x^{-4}}\;dx$$

3. Suppose that a cylindrical tank has height of 10 feet, base radius of 7 feet, and that the tank is half-full of water. The top of the tank is at ground level.

(a) Write an expression that approximates the work done in lifting a horizontal slice of water that is y feet below ground level to the ground's surface, given that the depth of the slice is Δy. Include appropriate units in your answer. (Assume that water weighs 64.5 lbs/ft^3.) *(Do not evaluate this integral.)*

Solution:

Let $0 \le y \le 5$. Choose a horizontal slice passing through (0, y) having thickness Δy.

Then the volume of this slice is $\pi\, 7^2 \Delta y$.

The work performed in moving this slice y units from the bottom of the tank is

$\Delta W = \pi\, 7^2 \Delta y\ (y)\ 64.5 = 49(64.5)\pi\,(10 - y)\ \Delta y$

(b) How much work is done to pump all of the water out? (Assume that water weighs 64.5 lbs/ft^3.) *(Do not evaluate this integral.)*

Solution:

Thus the total amount of work performed is given by

$$W = \int_5^{10} 49(64.5)\pi y \text{ ft-lbs}$$

4. Evaluate the following integral. Simplify.

$$\int \frac{x^2}{(x-2)(x-1)}dx$$

After long division, we have $\frac{x^2}{(x-2)(x-1)} = 1 + \frac{3x-2}{(x-2)(x-1)}$

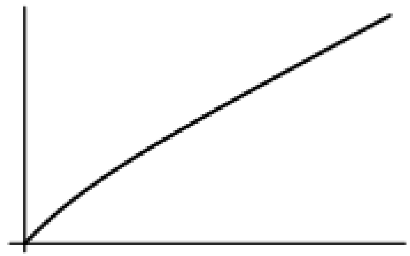
Using partial fractions; $\frac{3x-2}{(x-2)(x-1)} = \frac{4}{x-2} - \frac{1}{x-1}$

Hence the answer is $4\ \ln|x-2| - \ln|x-1| + C$

Extra Credit:

[14 points] An astute University of Michigan squirrel notes that the length of one strand of the web of a mathematically inclined spider is exactly $\int_0^6 \sqrt{2+2e^{-x}+e^{-2x}}\,dx$ cm, where x measures the horizontal distance from the wall of a campus building. The strand of web is shown in the figure to the right, below.

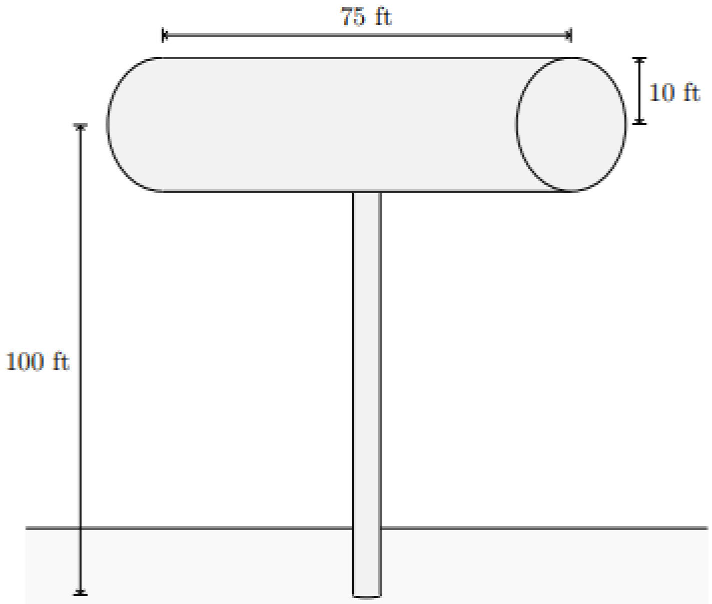
a. [7 points] Find an equation $y = f(x)$ that describes the shape of this strand of web.



Solution: We note that the squirrel is integrating to find the arclength of a curve. The arclength of a curve $y = f(x)$ is given by $\int_a^b \sqrt{1+(f'(x))^2}\,dx$, so we must have $(f'(x))^2 = 1 + 2e^{-x} + e^{-2x}$. Thus $f'(x) = \pm(1 + e^{-x})$. To match the graph shown we must take the positive sign, so $f(x) = x - e^{-x} + C$, for some constant C. Because the web starts at the origin $((x, y) = (0, 0))$, C must be 1, so $y = x - e^{-x} + 1$.

Extra Credit: De'von Baptiste is a shrewd industrialist. When energy costs are low, De'von pumps purified muck (which he gets for free from the city) into very tall tanks. In this way he stores cheap potential energy. Someday, when energy prices soar, Mr. Batiste will convert it all back into useful kinetic energy at a great profit. His tanks are cylinders 75 ft long with radius 10 ft. The center of a tank is 100 ft.

(a) What is the area, in square feet, of a cross-section parallel to the ground taken y feet above the center of the tank?



Solution: The cross-sections are rectangles with a length of 75 ft and a width $w(y)$ which depends on y. Using the Pythagorean Theorem we find that

$$10^2 = y^2 + (w(y)/2)^2 \longrightarrow w(y) = 2\sqrt{100 - y^2} = \sqrt{400 - 4y^2}.$$

Hence the area of a cross-section is

$$\text{Area of cross-section} = 75w(y) = 75 \cdot 2\sqrt{100 - y^2} = 150\sqrt{100 - y^2}.$$

(b) Write an integral which represents the total work (in foot-pounds) required to fill one of De'von Batiste's tanks with purified muck. Do not evaluate this integral.

Solution: If we consider the tank after its filled, we can compute the work required to get each slice of muck y feet above the center of the tank from the ground to its height at $100 + y$ ft above the ground. If $A(y) = 300\sqrt{100 - y^2}$ is the area of a cross-section y feet above the center of the tank, then the total work is

$$\begin{aligned}
\text{Total work} &= \int_{-10}^{10} (\text{density})(\text{distance})(\text{slice volume}) \\
&= \int_{-10}^{10} 800(100 + y)A(y)\, dy \\
&= \int_{-10}^{10} 800(100 + y)150\sqrt{100 - y^2}\, dy
\end{aligned}$$

I have hardly ever known a mathematician who was able to reason.

\- Stephen Hawking